Abstract

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Review of SLS 2018: Solovay reduction and continuity
- Introduction
- Solovay reduction and Lipschitz continuity
- From Lipschitz to Hölder
- Separation
- Summary of the former half

Ongoing research
- Sequences
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Characterization of quasi Solovay reduction via sequences

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- We would like to get a better understanding of the relationships between reduction and continuity.
- The former half: We review our talk at SLS 2018 (Sendai Logic School 2018). We introduced quasi Solovay reduction. We characterized it by existence of a certain real function that is Weihrauch-computable and Hölder continuous. We separated it from S-reduction and T-reduction.
- The latter half: We characterize quasi Solovay reduction via sequences. We invesitgate Solovay degrees of qS-complete reals.

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Left c.e. reals and Solovay reduction

Definition Suppose that α and β are reals.

1. α is left-c.e. if its left set (of Dedekind cut) is c.e.
2. α is Solovay reducible to β ($\alpha \leq_S \beta$)
   if $\exists$ a partial computable $f : \mathbb{Q} \to \mathbb{Q}$ $\exists d > 0$ $\forall q \in \mathbb{Q}$ (s.t. $q < \beta$), $f(q) \downarrow < \alpha$ and $|\alpha - f(q)| < d|\beta - q|$.

Fact If α and β are left-c.e.

$\exists f : (-\infty, \beta) \to (-\infty, \alpha)$ s.t.

- f is computable in the sense of Weihrauch (2000).
- $\{f(x) : x < \beta\}$ is cofinal in $(-\infty, \alpha)$.
- f is nondecreasing.

Take $\alpha_n \nearrow \alpha$, $\beta_n \nearrow \beta$. Make a line graph by (β_n, α_n).

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Lipschitz continuity

Suppose that $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$.

Definition

1. f is Lipschitz continuous if $\exists L > 0$ s.t. $\forall x_1, x_2 \in I \ |f(x_1) - f(x_2)| \leq L|x_1 - x_2|$.
2. f is Hölder continuous (with order ≤ 1) if there exists a positive real numbers H and $\xi \leq 1$ s.t. $\forall x_1, x_2 \in I \ |f(x_1) - f(x_2)| \leq H|x_1 - x_2|^{\xi}$.

The relationships between continuity concepts

If I is a bounded closed interval
analytic $\Rightarrow$ Lipschitz cont. $\Rightarrow$ Hölder cont. $\Rightarrow$ uniform cont.

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Characterization of Solovay red. via Lipschitz cont.

For left-c.e. reals α and β, we show the following.

Theorem 1 The following are equivalent.

1. $\alpha \leq_S \beta$
2. $\exists f : (-\infty, \beta) \to (-\infty, \alpha)$ s.t.
   (a) f is computable in the sense of Weihrauch (2000).
   (b) f is Lipschitz continuous in $(-\infty, \beta)$.
   (c) $\{f(x) : x < \beta\}$ is cofinal in $(-\infty, \alpha)$.
   (d) f is nondecreasing.

From Lipschitz to Hölder

We ask whether there exists a reducibility concept that corresponds to Hölder continuity (with order ≤ 1).

Answer is ~~almost~~ yes. ("almost" has been dropped, by Miyabe.)

Definition Suppose that α and β are reals.

α is quasi Solovay reducible to β ($\alpha \leq_{qS} \beta$) if $\exists$ a partial computable $f : \mathbb{Q} \to \mathbb{Q}$ $\exists d > 0$ $\exists \ell \in \mathbb{N}^+$ $\forall q \in \mathbb{Q}$ (s.t. $q < \beta$), $f(q) \downarrow < \alpha \wedge |\alpha - f(q)|^\ell < d|\beta - q|$.

Lemma 1

$\leq_{qS}$ satisfies reflexive law and transitive law. In addition, $\leq_{qS}$ is a standard reducibility (in particular, addition is a join for the degrees of left-c.e. reals).

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qS reduction and Hölder continuity

Theorem 2 **The following are equivalent.**

1. $\alpha \leq_{qS} \beta$
2. $\exists f : (-\infty, \beta) \to (-\infty, \alpha)$ s.t.
   (a) f is computable in the sense of Weihrauch (2000).
   (b′) f is Hölder continuous in $(-\infty, \beta)$.
   (c) $\{f(x) : x < \beta\}$ is cofinal in $(-\infty, \alpha)$.
   (d) f is nondecreasing.

As of SLS 2018, we had a weaker form of Theorem 2. Miyabe improved it.

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Separation of qS from other reductions

Lemma 2 Separation

Suppose that α and β are left-c.e. reals.

1. $\alpha \leq_S \beta \Rightarrow \not\Leftarrow \alpha \leq_{qS} \beta$
2. $\alpha \leq_{qS} \beta \Rightarrow \not\Leftarrow \alpha \leq_T \beta$

Summary of the former half

- Solovay reduction is characterized by existence of a certain real function that is computable (in the sense of Weihrauch) and Lipschitz continuous.
- We asked whether there exists a reducibility concept that corresponds to Hölder continuity.
- We introduced quasi Solovay reduction. Answer for the above question is ~~almost~~ yes.
- We separated qS reduction from Solovay reduction and Turing reduction.

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On going research

Characterization of qS red. via sequences

We are going to characterize quasi Solovay reduction via sequences. Suppose that α and β are left-c.e. reals.

Fact (Downey et al. 2002)

Suppose $r_n \nearrow \beta$ ($\{r_n\}$ computable). T. f. a. e.

1. $\alpha \leq_S \beta$
2. $\exists \{p_n\}$ (computable, $p_n \nearrow \alpha$) $\exists d > 0$ s. t. $\forall n \in \mathbb{N}\, p_n - p_{n-1} < d(r_n - r_{n-1})$.

Corollary (to Thm. 2) The fol. are equivalent.

1. $\alpha \leq_{qS} \beta$
2. $\exists \{p_n\}, \{r_n\}$ (computable, $p_n \nearrow \alpha$, $r_n \nearrow \beta$) $\exists d, \ell > 0$ s. t. $\forall n, m \in \mathbb{N}\, (n < m \to (p_m - p_n)^\ell < d(r_m - r_n))$.

qS-complete sets and partial randomness

Among left-c.e. reals, Solovay complete ⇔ 1-random. We investigate quasi Solovay complete real numbers.

Definition [Tadaki 2002] Let $\alpha \in \mathbb{R}$, $T \in (0, 1]$.

1. α is weakly Chaitin T-random if $\forall n \in \mathbb{N}^+[Tn \leq_+ K(\alpha \upharpoonright_n)]$.
2. α is T-compressible if $K(\alpha \upharpoonright_n) \leq Tn + o(n)$.
3. (Generalized halting probability) $\Omega^T := \sum_{p \in \mathrm{dom} U} 2^{-|p|/T}$

Fact [Tadaki 2002] (See also [Mayordomo 2002])

For each $T \in (0, 1]$, Ω^T is weakly Chaitin T-random, and T-compressible.

qS-complete sets and partial randomness

For each $T = 2^{-n}$ $(n \in \mathbb{N})$, we introduce a variant of Ω^T.

Def. Modified generalized halting probability Ω_T

1. Suppose that $0.a_1a_2a_3\cdots$ is a binary expansion of a real where 0 appear infinitely many often. $h_1(0.a_1a_2a_3\cdots) := 0.b_1b_2b_3\cdots$, where $b_{2n} = 1 - b_{2n-1}$ for each $n \geq 1$.
2. $\Omega_{2^0} := \Omega$, $\Omega_{2^{-(n+1)}} := h_1(\Omega_{2^{-n}})$

Lemma 3 qS-complete sets and partial randomness

Suppose $T = 2^{-n}$ and $n \in \mathbb{N}^+$.

1. Ω_T is weakly Chaitin T-random and T-compressible.
2. Ω_T is qS-complete among left-c.e.reals.

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Fact [Mayordomo 2002]

Given $\alpha \in 2^\omega$, the effective Hausdorff dimension $\dim(\alpha)$ is characterized as follows.

$$\dim(\alpha) = \liminf_n \frac{K(\alpha \upharpoonright n)}{n}$$

Suppose that qS-red. is a relation on left-c.e. sets.

Theorem 3 The fol. are equivalent for left-c.e. α.

1. α is qS-complete.
2. For some $n \in \mathbb{N}^+$, letting $T = 2^{-n}$, $\Omega_T \leq_S \alpha$.
3. $\dim(\alpha) > 0$

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Fact [Miyabe Nies Stephan 2018]

For each rational $r \in (0,1)$, the set F_r of Solovay degrees $\mathbf{a}$ s. t. $\dim \mathbf{a} > r$ is a filter in Solovay degrees.

- $\mathbf{a} \in F_r \land \mathbf{a} \leq \mathbf{b} \ \Rightarrow \ \mathbf{b} \in F_r$
- $\mathbf{a}, \mathbf{b} \in F_r \ \Rightarrow \ \exists \mathbf{c} \in F_r \ \mathbf{c} \leq \mathbf{a} \land \mathbf{c} \leq \mathbf{b}$

Corollary (to Thm. 3)

1. The Solovay degrees of all qS-complete left-c.e. reals is a filter in Solovay degrees.
2. qS-complete $\Rightarrow$ not 1-generic

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Thank you for your attention.

We have uploaded our preprint on the former half.

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