Ideal-versions of Bolzano-Weierstrass property

Jiakui Yu，Shuguo Zhang

College of Mathematics
Sichuan University

The 9th International Conference on Computability Theory and Foundations of Mathematics
Wu Han, March 21-27, 2019

Ideals on ω

Let S be a set and $\mathcal{I}$ be a collection of subsets of S which contains $\emptyset$ and does not contain S.

Definition

$\mathcal{I}$ is called an *ideal* if it is closed under taking subsets and finite unions.

Usually, we suppose $S = \omega$ and the ideal containing all finite sets. Actually, we can think ideal as collection of small subsets.

Ideals on ω

Let S be a set and $\mathcal{I}$ be a collection of subsets of S which contains $\emptyset$ and does not contain S.

Definition

$\mathcal{I}$ is called an *ideal* if it is closed under taking subsets and finite unions.

Usually, we suppose $S = \omega$ and the ideal containing all finite sets. Actually, we can think ideal as collection of small subsets.

Ideals on ω

Let S be a set and $\mathcal{I}$ be a collection of subsets of S which contains $\emptyset$ and does not contain S.

Definition

$\mathcal{I}$ is called an *ideal* if it is closed under taking subsets and finite unions.

Usually, we suppose $S = \omega$ and the ideal containing all finite sets. Actually, we can think ideal as collection of small subsets.

Let $\mathcal{I}$ be an ideal on ω, the following notations will be used frequently.

- $\mathcal{I}^+ = \{A \subseteq \omega : A \notin \mathcal{I}\}$;
- $\mathcal{I}^* = \{A \subseteq \omega : \omega \setminus A \in \mathcal{I}\}$;
- $\mathcal{I}|A = \{I \cap A : I \in \mathcal{I}\}$, for each $A \in \mathcal{I}^+$,

If $A \in \mathcal{I}^+$, we say that A is an $\mathcal{I}$-**positive set**.

Let $\mathcal{I}$ be an ideal on ω, the following notations will be used frequently.

- $\mathcal{I}^+ = \{A \subseteq \omega : A \notin \mathcal{I}\}$;
- $\mathcal{I}^* = \{A \subseteq \omega : \omega \setminus A \in \mathcal{I}\}$;
- $\mathcal{I}|A = \{I \cap A : I \in \mathcal{I}\}$, for each $A \in \mathcal{I}^+$,

If $A \in \mathcal{I}^+$, we say that A is an $\mathcal{I}$-**positive set**.

Ideals with combinational properties

The following special ideals were studied in set theory, topology and combinatorics:

Definition

- $\mathcal{I}$ is *local Q* if for every partition $\{A_n : n \in \omega\} \subset Fin$ of ω, there exists $A \in \mathcal{I}^+$ such that $|A \cap A_n| \leq 1$ for each $n \in \omega$;
- $\mathcal{I}$ is *locally selective* if for every partition $\{A_n : n \in \omega\} \subset \mathcal{I}$ of ω, there exists $A \in \mathcal{I}^+$ such that $|A \cap A_n| \leq 1$ for each $n \in \omega$.
- $\mathcal{I}$ is *weak Q* if for every $A \in \mathcal{I}^+$, $\mathcal{I}|A$ is local Q.
- $\mathcal{I}$ is *weakly selective* if for every $A \in \mathcal{I}^+$, $\mathcal{I}|A$ is locally selective.

Ideals with combinational properties

Definition

Let $\mathcal{I}$ be an ideal on ω, $r \in \omega$, and $c : [\omega]^2 \to \{0, \cdots, r-1\}$ being a coloring. $A \subset \omega$ is *$\mathcal{I}$-homogeneous* for c if there is $k \in \{0, \cdots, r-1\}$ such that for every $a \in A$,

$$\{b \in A : c(\{a, b\}) \neq k\} \in \mathcal{I}.$$

Definition

Let $\mathcal{I}$ be an ideal on ω. $\mathcal{I}$ is *Ramsey** if for every finite coloring of $[\omega]^2$ there exists an $\mathcal{I}$-homogeneous $A \in \mathcal{I}^+$.

Ideals with combinational properties

Definition

Let $\mathcal{I}$ be an ideal on ω, $r \in \omega$, and $c : [\omega]^2 \to \{0, \cdots, r-1\}$ being a coloring. $A \subset \omega$ is *$\mathcal{I}$-homogeneous* for c if there is $k \in \{0, \cdots, r-1\}$ such that for every $a \in A$,

$$\{b \in A : c(\{a, b\}) \neq k\} \in \mathcal{I}.$$

Definition

Let $\mathcal{I}$ be an ideal on ω. $\mathcal{I}$ is $Ramsey^*$ if for every finite coloring of $[\omega]^2$ there exists an $\mathcal{I}$-homogeneous $A \in \mathcal{I}^+$.

Ideals with combinational properties

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω. We say that the pair *$(\mathcal{I},\mathcal{J})$ is $Ramsey^*$* if for every finite coloring of $[\omega]^2$ there exists $A \in \mathcal{I}^+$ that is $\mathcal{J}$-homogeneous.

When $\mathcal{I}=\mathcal{J}$ we say that $\mathcal{I}$ has $Ramsey^*$ instead of $(\mathcal{I},\mathcal{I})$ having $Ramsey^*$. It is not hard to see that for any ideals $\mathcal{I}$, $\mathcal{J}$ on ω, if $\mathcal{I}\not\subset\mathcal{J}$, then the pair $(\mathcal{J},\mathcal{I})$ is $Ramsey^*$.

Ideals with combinational properties

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω. We say that the pair *$(\mathcal{I},\mathcal{J})$ is $Ramsey^*$* if for every finite coloring of $[\omega]^2$ there exists $A \in \mathcal{I}^+$ that is $\mathcal{J}$-homogeneous.

When $\mathcal{I}=\mathcal{J}$ we say that $\mathcal{I}$ has $Ramsey^*$ instead of $(\mathcal{I},\mathcal{I})$ having $Ramsey^*$. It is not hard to see that for any ideals $\mathcal{I}$, $\mathcal{J}$ on ω, if $\mathcal{I} \not\subset \mathcal{J}$, then the pair $(\mathcal{J},\mathcal{I})$ is $Ramsey^*$.

Ideals with combinational properties

Let $\mathcal{I}$ be an ideal on ω. Recall that a sequence $\langle x_n : n \in A\rangle$ in $[0,1]$ is *$\mathcal{I}$-increasing* if for every $N \in A$

$$\{n \in A : x_N \geq x_n\} \in \mathcal{I}.$$

Analogously, we can define $\mathcal{I}$-decreasing, $\mathcal{I}$-nonincreasing and $\mathcal{I}$-nondecreasing sequences. A sequence $\langle x_n : n \in \omega\rangle$ in $[0,1]$ is $\mathcal{I}$-monotone if it is $\mathcal{I}$-nonincreasing or $\mathcal{I}$-nondecreasing.

Definition

Let $\mathcal{I}$ be an ideal on ω, we say that $\mathcal{I}$ is Mon^* if for every sequence $\langle x_n : n \in \omega\rangle$ in $[0,1]$ there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{I}$-monotone.

Ideals with combinational properties

Let $\mathcal{I}$ be an ideal on ω. Recall that a sequence $\langle x_n : n \in A\rangle$ in $[0,1]$ is *$\mathcal{I}$-increasing* if for every $N \in A$

$$\{n \in A : x_N \geq x_n\} \in \mathcal{I}.$$

Analogously, we can define $\mathcal{I}$-decreasing, $\mathcal{I}$-nonincreasing and $\mathcal{I}$-nondecreasing sequences. A sequence $\langle x_n : n \in \omega\rangle$ in $[0,1]$ is $\mathcal{I}$-monotone if it is $\mathcal{I}$-nonincreasing or $\mathcal{I}$-nondecreasing.

Definition

Let $\mathcal{I}$ be an ideal on ω, we say that $\mathcal{I}$ is Mon^* if for every sequence $\langle x_n : n \in \omega\rangle$ in $[0,1]$ there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{I}$-monotone.

Ideals with combinational properties

Let $\mathcal{I}$ be an ideal on ω. Recall that a sequence $\langle x_n : n \in A\rangle$ in $[0,1]$ is *$\mathcal{I}$-increasing* if for every $N \in A$

$$\{n \in A : x_N \geq x_n\} \in \mathcal{I}.$$

Analogously, we can define $\mathcal{I}$-decreasing, $\mathcal{I}$-nonincreasing and $\mathcal{I}$-nondecreasing sequences. A sequence $\langle x_n : n \in \omega\rangle$ in $[0,1]$ is $\mathcal{I}$-monotone if it is $\mathcal{I}$-nonincreasing or $\mathcal{I}$-nondecreasing.

Definition

Let $\mathcal{I}$ be an ideal on ω, we say that $\mathcal{I}$ is Mon^* if for every sequence $\langle x_n : n \in \omega\rangle$ in $[0,1]$ there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{I}$-monotone.

Ideals with combinational properties

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω. We say that the pair *$(\mathcal{I},\mathcal{J})$ is Mon^** if every sequence in $[0,1]$ contains a $\mathcal{J}$-monotone $\mathcal{I}$-subsequence. That is, for every sequence $\langle x_n : n\in\omega\rangle$ in $[0,1]$, there exists $A\in\mathcal{I}^+$ such that $\langle x_n : n\in A\rangle$ is $\mathcal{J}$-monotone.

Ideals with combinational properties

Let $\mathcal{I}$ be an ideal on ω. Recall that $\mathcal{I}$ is *dense* (or tall) if every infinite set $A \subseteq \omega$ contains an infinite subset B that belongs to $\mathcal{I}$.

Definition

Let $\mathcal{A}$, $\mathcal{B}$ be sets of subsets of ω. We say that *$\mathcal{B}$ is $\mathcal{A}$-dense* if for each $A \in \mathcal{A}$, there exists an infinite $B \subseteq A$ such that $B \in \mathcal{B}$.

Evidently, $\mathcal{I}$ being $[\omega]^\omega$-dense coincides with $\mathcal{I}$ being dense. In addition, for any ideal $\mathcal{I}$, $\mathcal{I}^+$ is $[\omega]^\omega$-dense if, and only if $\mathcal{I} = Fin$.

Ideals with combinational properties

Let $\mathcal{I}$ be an ideal on ω. Recall that $\mathcal{I}$ is *dense* (or tall) if every infinite set $A \subseteq \omega$ contains an infinite subset B that belongs to $\mathcal{I}$.

Definition

Let $\mathcal{A}$, $\mathcal{B}$ be sets of subsets of ω. We say that *$\mathcal{B}$ is $\mathcal{A}$-dense* if for each $A \in \mathcal{A}$, there exists an infinite $B \subseteq A$ such that $B \in \mathcal{B}$.

Evidently, $\mathcal{I}$ being $[\omega]^\omega$-dense coincides with $\mathcal{I}$ being dense. In addition, for any ideal $\mathcal{I}$, $\mathcal{I}^+$ is $[\omega]^\omega$-dense if, and only if $\mathcal{I} = Fin$.

Ideals with combinational properties

Let $\mathcal{I}$ be an ideal on ω. Recall that $\mathcal{I}$ is *dense* (or tall) if every infinite set $A \subseteq \omega$ contains an infinite subset B that belongs to $\mathcal{I}$.

Definition

Let $\mathcal{A}$, $\mathcal{B}$ be sets of subsets of ω. We say that *$\mathcal{B}$ is $\mathcal{A}$-dense* if for each $A \in \mathcal{A}$, there exists an infinite $B \subseteq A$ such that $B \in \mathcal{B}$.

Evidently, $\mathcal{I}$ being $[\omega]^\omega$-dense coincides with $\mathcal{I}$ being dense. In addition, for any ideal $\mathcal{I}$, $\mathcal{I}^+$ is $[\omega]^\omega$-dense if, and only if $\mathcal{I} = Fin$.

Ideals with combinational properties

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω. For a map $\varphi : \omega \to \omega$, the image of $\mathcal{J}$ is defined by

$$\varphi(\mathcal{J}) = \{A \subseteq \omega : \varphi^{-1}(A) \in \mathcal{J}\}.$$

Clearly, $\varphi(\mathcal{J})$ is closed under subsets and finite unions and $\omega \notin \varphi(\mathcal{J})$. Moreover, if φ is finite-to-one then $\varphi(\mathcal{J})$ is an ideal.

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω,

- $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $\varphi : \omega \to \omega$ such that $\mathcal{I} \subseteq \varphi(\mathcal{J})$, i.e, $\varphi^{-1}(A) \in \mathcal{J}$ for any $A \in \mathcal{I}$;
- $\mathcal{I} \leq_{KB} \mathcal{J}$ if there is a finite-to-one function $\varphi : \omega \to \omega$ such that $\mathcal{I} \leq_K \mathcal{J}$;
- $\mathcal{I} \leq_{RB} \mathcal{J}$ if there is a finite-to-one function $\varphi : \omega \to \omega$ such that $A \in \mathcal{I}$ if, and only if $\varphi^{-1}(A) \in \mathcal{J}$ for every $A \subset \omega$;

Ideals with combinational properties

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω. For a map $\varphi:\omega\to\omega$, the image of $\mathcal{J}$ is defined by

$$\varphi(\mathcal{J})=\{A\subseteq\omega:\varphi^{-1}(A)\in\mathcal{J}\}.$$

Clearly, $\varphi(\mathcal{J})$ is closed under subsets and finite unions and $\omega\notin\varphi(\mathcal{J})$. Moreover, if φ is finite-to-one then $\varphi(\mathcal{J})$ is an ideal.

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω,

- $\mathcal{I}\leq_K\mathcal{J}$ if there is a function $\varphi:\omega\to\omega$ such that $\mathcal{I}\subseteq\varphi(\mathcal{J})$, i.e, $\varphi^{-1}(A)\in\mathcal{J}$ for any $A\in\mathcal{I}$;
- $\mathcal{I}\leq_{KB}\mathcal{J}$ if there is a finite-to-one function $\varphi:\omega\to\omega$ such that $\mathcal{I}\leq_K\mathcal{J}$;
- $\mathcal{I}\leq_{RB}\mathcal{J}$ if there is a finite-to-one function $\varphi:\omega\to\omega$ such that $A\in\mathcal{I}$ if, and only if $\varphi^{-1}(A)\in\mathcal{J}$ for every $A\subset\omega$;

Ideals with combinational properties

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω. For a map $\varphi:\omega\to\omega$, the image of $\mathcal{J}$ is defined by

$$\varphi(\mathcal{J})=\{A\subseteq\omega:\varphi^{-1}(A)\in\mathcal{J}\}.$$

Clearly, $\varphi(\mathcal{J})$ is closed under subsets and finite unions and $\omega\notin\varphi(\mathcal{J})$. Moreover, if φ is finite-to-one then $\varphi(\mathcal{J})$ is an ideal.

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω,

- $\mathcal{I}\leq_K\mathcal{J}$ if there is a function $\varphi:\omega\to\omega$ such that $\mathcal{I}\subseteq\varphi(\mathcal{J})$, i.e, $\varphi^{-1}(A)\in\mathcal{J}$ for any $A\in\mathcal{I}$;
- $\mathcal{I}\leq_{KB}\mathcal{J}$ if there is a finite-to-one function $\varphi:\omega\to\omega$ such that $\mathcal{I}\leq_K\mathcal{J}$;
- $\mathcal{I}\leq_{RB}\mathcal{J}$ if there is a finite-to-one function $\varphi:\omega\to\omega$ such that $A\in\mathcal{I}$ if, and only if $\varphi^{-1}(A)\in\mathcal{J}$ for every $A\subset\omega$;

Ideal-convergence

Let $\mathcal{I}$ be an ideal on ω, and X being a topological space. For sequence $\langle x_n : n \in \omega \rangle$ in X, we say that $\langle x_n : n \in \omega \rangle$ is *$\mathcal{I}$-convergent* to l if for each open neighborhood U of l,

$$\{n : x_n \notin U\} \in \mathcal{I}.$$

The notion of $\mathcal{I}$-convergence is a generalization of the classical one. It was first considered by Steinhaus and Fast in the case of the ideal of sets of statistical density 0:

$$\mathcal{I}_d = \{A \subset \omega : limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0\}.$$

Ideal-convergence

Let $\mathcal{I}$ be an ideal on ω, and X being a topological space. For sequence $\langle x_n : n \in \omega\rangle$ in X, we say that $\langle x_n : n \in \omega\rangle$ is *$\mathcal{I}$-convergent* to l if for each open neighborhood U of l,

$$\{n : x_n \notin U\} \in \mathcal{I}.$$

The notion of $\mathcal{I}$-convergence is a generalization of the classical one. It was first considered by Steinhaus and Fast in the case of the ideal of sets of statistical density 0:

$$\mathcal{I}_d = \{A \subset \omega : limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0\}.$$

Ideal-convergence

By an $\mathcal{I}$-subsequence of $\langle x_n : n \in \omega \rangle$ we means $\langle x_n : n \in A \rangle$ for some $A \notin \mathcal{I}$. Filipów, Mrożek, Recław and Szuca introduced the following notions.

Definition

Let $\mathcal{I}$ be an ideal on ω, X being a topological space.

- $(X, \mathcal{I})$ satisfies BW if every sequence in X has $\mathcal{I}$-convergent $\mathcal{I}$-subsequence;
- $(X, \mathcal{I})$ satisfies $FinBW$ if every sequence in X has convergent $\mathcal{I}$-subsequence;

Ideal-convergence

By an $\mathcal{I}$-subsequence of $\langle x_n : n \in \omega \rangle$ we means $\langle x_n : n \in A \rangle$ for some $A \notin \mathcal{I}$. Filipów, Mrożek, Recław and Szuca introduced the following notions.

Definition

Let $\mathcal{I}$ be an ideal on ω, X being a topological space.

- $(X, \mathcal{I})$ satisfies BW if every sequence in X has $\mathcal{I}$-convergent $\mathcal{I}$-subsequence;
- $(X, \mathcal{I})$ satisfies $FinBW$ if every sequence in X has convergent $\mathcal{I}$-subsequence;

What will we consider?

We mainly consider the following questions:

Question

These notions involve two ideals: $\mathcal{I}$ and Fin. We are interested in the question how about if we replace Fin by another ideal $\mathcal{J}$?

Here is the key definition, which is a common generalization of these types.

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, X being a topological space. We say that X has $(\mathcal{I},\mathcal{J})$-$BW$ property if every sequence in X has $\mathcal{J}$-convergent $\mathcal{I}$-subsequence.

It is worthy to point out that if $\mathcal{I}\not\subseteq\mathcal{J}$, then for arbitrary space X, it has $(\mathcal{J},\mathcal{I})$-$BW$ property. Indeed, picking $A\in\mathcal{I}\setminus\mathcal{J}$, A can deal with any sequence in X.

What will we consider?

We mainly consider the following questions:

Question

These notions involve two ideals: $\mathcal{I}$ and Fin. We are interested in the question how about if we replace Fin by another ideal $\mathcal{J}$?

Here is the key definition, which is a common generalization of these types.

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, X being a topological space. We say that X has $(\mathcal{I},\mathcal{J})$-$BW$ property if every sequence in X has $\mathcal{J}$-convergent $\mathcal{I}$-subsequence.

It is worthy to point out that if $\mathcal{I} \not\subseteq \mathcal{J}$, then for arbitrary space X, it has $(\mathcal{J},\mathcal{I})$-$BW$ property. Indeed, picking $A \in \mathcal{I} \setminus \mathcal{J}$, A can deal with any sequence in X.

What will we consider?

We mainly consider the following questions:

Question

These notions involve two ideals: $\mathcal{I}$ and Fin. We are interested in the question how about if we replace Fin by another ideal $\mathcal{J}$?

Here is the key definition, which is a common generalization of these types.

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, X being a topological space. We say that X has $(\mathcal{I},\mathcal{J})$-$BW$ property if every sequence in X has $\mathcal{J}$-convergent $\mathcal{I}$-subsequence.

It is worthy to point out that if $\mathcal{I} \not\subseteq \mathcal{J}$, then for arbitrary space X, it has $(\mathcal{J},\mathcal{I})$-$BW$ property. Indeed, picking $A \in \mathcal{I} \setminus \mathcal{J}$, A can deal with any sequence in X.

$(\mathcal{I},\mathcal{J})$-splitting family

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, and $\mathcal{S} \subset [\omega]^\omega$. We say that $\mathcal{S}$ is an $(\mathcal{I},\mathcal{J})$-splitting family if for every $A \in \mathcal{I}^+$ there exists $X \in \mathcal{S}$ such that both of $A \cap X$ and $A \setminus X$ belong to $\mathcal{J}^+$.

Evidently, when $\mathcal{I}$ is equal to $\mathcal{J}$, the $(\mathcal{I},\mathcal{J})$-splitting family coincides with the $\mathcal{I}$-splitting family:

Definition

Let $\mathcal{S} \subseteq [\omega]^\omega$, and $\mathcal{I}$ being an ideal on ω. A family $\mathcal{S}$ is $\mathcal{I}$-splitting if for every $A \in \mathcal{I}^+$ there exists $S \in \mathcal{S}$ such that $A \cap S \in \mathcal{I}^+$ and $A \setminus S \in \mathcal{I}^+$.

$(\mathcal{I},\mathcal{J})$-splitting family

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, and $\mathcal{S} \subset [\omega]^\omega$. We say that $\mathcal{S}$ is an $(\mathcal{I},\mathcal{J})$-splitting family if for every $A \in \mathcal{I}^+$ there exists $X \in \mathcal{S}$ such that both of $A \cap X$ and $A \setminus X$ belong to $\mathcal{J}^+$.

Evidently, when $\mathcal{I}$ is equal to $\mathcal{J}$, the $(\mathcal{I},\mathcal{J})$-splitting family coincides with the $\mathcal{I}$-splitting family:

Definition

Let $\mathcal{S} \subseteq [\omega]^\omega$, and $\mathcal{I}$ being an ideal on ω. A family $\mathcal{S}$ is $\mathcal{I}$-splitting if for every $A \in \mathcal{I}^+$ there exists $S \in \mathcal{S}$ such that $A \cap S \in \mathcal{I}^+$ and $A \setminus S \in \mathcal{I}^+$.

$(\mathcal{I},\mathcal{J})$-splitting family

Definition

Let $\mathfrak{s}(\mathcal{I},\mathcal{J})$ be the smallest cardinality of an $(\mathcal{I},\mathcal{J})$-splitting family.

It is easy to see that the $\mathfrak{s}(Fin, Fin)$ is just the *splitting number* $\mathfrak{s}$ introduced and $\mathfrak{s}(\mathcal{I},\mathcal{I})$ is just $\mathfrak{s}(\mathcal{I})$.

Theorem (Filipów, Mrożek, Recław and Szuca)

$\mathcal{I}$ satisfies BW if, and only if $\mathfrak{s}(\mathcal{I}) > \omega$

$(\mathcal{I},\mathcal{J})$-splitting family

Definition

Let $\mathfrak{s}(\mathcal{I},\mathcal{J})$ be the smallest cardinality of an $(\mathcal{I},\mathcal{J})$-splitting family.

It is easy to see that the $\mathfrak{s}(Fin, Fin)$ is just the *splitting number* $\mathfrak{s}$ introduced and $\mathfrak{s}(\mathcal{I},\mathcal{I})$ is just $\mathfrak{s}(\mathcal{I})$.

Theorem (Filipów, Mrożek, Recław and Szuca)

$\mathcal{I}$ satisfies BW if, and only if $\mathfrak{s}(\mathcal{I}) > \omega$

$(\mathcal{I},\mathcal{J})$-splitting family

Definition

Let $\mathfrak{s}(\mathcal{I},\mathcal{J})$ be the smallest cardinality of an $(\mathcal{I},\mathcal{J})$-splitting family.

It is easy to see that the $\mathfrak{s}(Fin, Fin)$ is just the *splitting number* $\mathfrak{s}$ introduced and $\mathfrak{s}(\mathcal{I},\mathcal{I})$ is just $\mathfrak{s}(\mathcal{I})$.

Theorem (Filipów, Mrożek, Recław and Szuca)

$\mathcal{I}$ satisfies BW if, and only if $\mathfrak{s}(\mathcal{I}) > \omega$

$(\mathcal{I},\mathcal{J})$-small set

Let $r\in\omega$, $s\in r^n$ and $i\in\{0,\cdots,r-1\}$, by $s\frown i$ we mean the sequence of length $n+1$ (write $lh(s)=n+1$) which extends s by i. If $x\in r^\omega$ and $n\in\omega$, $x|n$ denotes the initial segment $x|n=\langle x(0),x(1),\cdots,x(n-1)\rangle$.

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω. $A\subset\omega$ is called an $(\mathcal{I},\mathcal{J})$-small set if there exists $r\in\omega$, and exists a family $\{A_s : s\in r^{<\omega}\}$ such that for all $s\in r^{<\omega}$, we have

- S_1 $A_\emptyset=A$,
- S_2 $A_s=A_{s\frown 0}\cup\cdots\cup A_{s\frown(r-1)}$,
- S_3 $A_{s\frown i}\cap A_{s\frown j}=\emptyset$ for every $i\neq j$,
- S_4 for every $b\in r^\omega$, every $X\subset\omega$, if $X\setminus A_{b|n}\in\mathcal{I}$ for each $n\in\omega$, then $X\in\mathcal{J}$.

$(\mathcal{I},\mathcal{J})$-small set

Let $r \in \omega$, $s \in r^n$ and $i \in \{0, \cdots, r-1\}$, by $s \frown i$ we mean the sequence of length $n+1$ (write $lh(s) = n+1$) which extends s by i. If $x \in r^\omega$ and $n \in \omega$, $x|n$ denotes the initial segment $x|n = \langle x(0), x(1), \cdots, x(n-1)\rangle$.

Definition

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω. $A \subset \omega$ is called an $(\mathcal{I},\mathcal{J})$-small set if there exists $r \in \omega$, and exists a family $\{A_s : s \in r^{<\omega}\}$ such that for all $s \in r^{<\omega}$, we have

- S_1 $A_\emptyset = A$,
- S_2 $A_s = A_{s\frown 0} \cup \cdots \cup A_{s\frown(r-1)}$,
- S_3 $A_{s\frown i} \cap A_{s\frown j} = \emptyset$ for every $i \neq j$,
- S_4 for every $b \in r^\omega$, every $X \subset \omega$, if $X \setminus A_{b|n} \in \mathcal{I}$ for each $n \in \omega$, then $X \in \mathcal{J}$.

$(\mathcal{I},\mathcal{J})$-small set

Definition

Let $\mathcal{S}_{(\mathcal{I},\mathcal{J})}$ denote all $(\mathcal{I},\mathcal{J})$-small sets in $\mathcal{P}(\omega)$.

Note that $\mathcal{S}_{(\mathcal{I},\mathcal{J})} \neq \emptyset$ if, and only if $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{S}_{(\mathcal{I},\mathcal{J})}$.

$(\mathcal{I},\mathcal{J})$-small set

Definition

Let $\mathcal{S}_{(\mathcal{I},\mathcal{J})}$ denote all $(\mathcal{I},\mathcal{J})$-small sets in $\mathcal{P}(\omega)$.

Note that $\mathcal{S}_{(\mathcal{I},\mathcal{J})} \neq \emptyset$ if, and only if $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{S}_{(\mathcal{I},\mathcal{J})}$.

$(\mathcal{I},\mathcal{J})$-small set

Definition

Let $\mathcal{S}_{(\mathcal{I},\mathcal{J})}$ denote all $(\mathcal{I},\mathcal{J})$-small sets in $\mathcal{P}(\omega)$.

Note that $\mathcal{S}_{(\mathcal{I},\mathcal{J})} \neq \emptyset$ if, and only if $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{S}_{(\mathcal{I},\mathcal{J})}$.

Our results and these sketch of proofs

Theorem

$\omega \notin \mathcal{S}_{(\mathcal{I},\mathcal{J})}$ *if, and only if* $[0,1]$ *satisfies* $(\mathcal{J},\mathcal{I})$-$BW$.

Sketch of proof

The key fact:

Lemma

$(\mathcal{J},\mathcal{I})$-BW property is preserved for closed subsets and continuous images.

Thus, we consider the Cantor space 2^ω instead of $[0,1]$. Assume that $\omega \notin \mathcal{S}_{(\mathcal{I},\mathcal{J})}$. For every sequence $\langle x_n : n \in \omega \rangle$ in 2^ω, every $s \in 2^{<\omega}$, put

$$A_s = \{n : s \subset x_n\}.$$

Then $\{A_s : s \in 2^{<\omega}\}$ satisfies $S_1 - S_3$. Since $\omega \notin \mathcal{S}_{(\mathcal{I},\mathcal{J})}$, by the condition S_4, there exists $X \notin \mathcal{J}$ and $b \in 2^\omega$ such that $X \setminus A_{b|n} \in \mathcal{I}$ for each $n \in \omega$. Then $\langle x_n : n \in X \rangle$ is $\mathcal{I}$-convergent to b.

Sketch of proof

The key fact:

Lemma

$(\mathcal{J},\mathcal{I})$-BW property is preserved for closed subsets and continuous images.

Thus, we consider the Cantor space 2^ω instead of $[0,1]$. Assume that $\omega \notin \mathcal{S}_{(\mathcal{I},\mathcal{J})}$. For every sequence $\langle x_n : n \in \omega \rangle$ in 2^ω, every $s \in 2^{<\omega}$, put

$$A_s = \{n : s \subset x_n\}.$$

Then $\{A_s : s \in 2^{<\omega}\}$ satisfies $S_1 - S_3$. Since $\omega \notin \mathcal{S}_{(\mathcal{I},\mathcal{J})}$, by the condition S_4, there exists $X \notin \mathcal{J}$ and $b \in 2^\omega$ such that $X \setminus A_{b|n} \in \mathcal{I}$ for each $n \in \omega$. Then $\langle x_n : n \in X \rangle$ is $\mathcal{I}$-convergent to b.

Sketch of proof

The key fact:

Lemma

$(\mathcal{J},\mathcal{I})$-BW property is preserved for closed subsets and continuous images.

Thus, we consider the Cantor space 2^ω instead of $[0,1]$. Assume that $\omega \notin \mathcal{S}_{(\mathcal{I},\mathcal{J})}$. For every sequence $\langle x_n : n \in \omega\rangle$ in 2^ω, every $s \in 2^{<\omega}$, put

$$A_s = \{n : s \subset x_n\}.$$

Then $\{A_s : s \in 2^{<\omega}\}$ satisfies $S_1 - S_3$. Since $\omega \notin \mathcal{S}_{(\mathcal{I},\mathcal{J})}$, by the condition S_4, there exists $X \notin \mathcal{J}$ and $b \in 2^\omega$ such that $X \setminus A_{b|n} \in \mathcal{I}$ for each $n \in \omega$. Then $\langle x_n : n \in X\rangle$ is $\mathcal{I}$-convergent to b.

Sketch of proof

Suppose that $\omega \in \mathcal{S}_{(\mathcal{I},\mathcal{J})}$. So there exists $r \in \omega$, $\{A_s : s \in r^{<\omega}\}$ such that the conditions S_1-S_4 are fulfilled. Note that for each $n \in \omega$, there is exactly one $x_n \in 2^\omega$ such that $n \in A_{x_n|l}$ for each $l \in \omega$. Then we obtain a sequence $\langle x_n : n \in \omega \rangle$ in 2^ω. Since 2^ω satisfies $(\mathcal{J},\mathcal{I})$-BW, the sequence has an $\mathcal{I}$-convergent $\mathcal{J}$-subsequence, namely, there is a $x \in 2^\omega$ and $X \subseteq \omega$ with $X \in \mathcal{J}^+$ such that $\langle x_n : n \in X \rangle$ is $\mathcal{I}$-convergent to x. Since for each $l \in \omega$

$$X \setminus A_{x|l} \subseteq \{n \in X : |x - x_n| \geq \tfrac{1}{2^l}\} \in \mathcal{I}.$$

By the condition S_4, $X \in \mathcal{J}$, but this contradicts the fact that $X \in \mathcal{J}^+$. Therefore, we complete the proof.

Theorem

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω with $\mathcal{J}\subseteq\mathcal{I}$. In the following list of conditions each implies the next.

(1) $\mathfrak{s}(\mathcal{I},\mathcal{J})>\omega$.

(2) $[0,1]$ *satisfies* $(\mathcal{I},\mathcal{J})$-$BW$.

(3) $\mathfrak{s}(\mathcal{J},\mathcal{I})>\omega$.

Sketch of proof

$(1) \Rightarrow (2)$ Suppose that $[0,1]$ does not have $(\mathcal{I},\mathcal{J})$-$BW$. By Theorem 3.4, ω is a $(\mathcal{J},\mathcal{I})$-small set. We may assume that there exists a $r \in \omega$, and a family $\{A_s : s \in r^{<\omega}\}$ such that the conditions $S_1 - S_3$ are fulfilled. In what follows we will show that $\{A_s : s \in r^{<\omega}\}$ is an $(\mathcal{I},\mathcal{J})$-splitting family. For the sake of contradiction, suppose that there is $X \in \mathcal{I}^+$ such that for every $s \in r^{<\omega}$ either $X \cap A_s \in \mathcal{J}$ or $X \setminus A_s \in \mathcal{J}$. Put

$$T = \{s \in r^{<\omega} : X \setminus A_s \in \mathcal{J}\}.$$

Then T is a tree on $\{0, \cdots, r-1\}$ with finite branches for every level. In order to see that T is an infinite tree, we need the following lemma:

Lemma

For any $n \in \omega$, there is $s \in r^n$ such that $X \setminus A_s \in \mathcal{J}$.

Sketch of proof

$(1) \Rightarrow (2)$ Suppose that $[0,1]$ does not have $(\mathcal{I},\mathcal{J})$-$BW$. By Theorem 3.4, ω is a $(\mathcal{J},\mathcal{I})$-small set. We may assume that there exists a $r \in \omega$, and a family $\{A_s : s \in r^{<\omega}\}$ such that the conditions $S_1 - S_3$ are fulfilled. In what follows we will show that $\{A_s : s \in r^{<\omega}\}$ is an $(\mathcal{I},\mathcal{J})$-splitting family. For the sake of contradiction, suppose that there is $X \in \mathcal{I}^+$ such that for every $s \in r^{<\omega}$ either $X \cap A_s \in \mathcal{J}$ or $X \setminus A_s \in \mathcal{J}$. Put

$$T = \{s \in r^{<\omega} : X \setminus A_s \in \mathcal{J}\}.$$

Then T is a tree on $\{0,\cdots,r-1\}$ with finite branches for every level. In order to see that T is an infinite tree, we need the following lemma:

Lemma

For any $n \in \omega$, there is $s \in r^n$ such that $X \setminus A_s \in \mathcal{J}$.

Sketch of proof

Since T is an infinite tree with finite branches, by König's lemma, there exists $b \in r^\omega$ such that $X \setminus A_{b|n} \in \mathcal{J}$ for every $n \in \omega$. According to the fact that ω is an $(\mathcal{J},\mathcal{I})$-small set we have that $X \in \mathcal{I}$. Contradiction.

(2) $\Rightarrow$ (3) Suppose that $\mathfrak{s}(\mathcal{J},\mathcal{I}) = \omega$, and $\{S_n : n \in \omega\}$ be a $(\mathcal{J},\mathcal{I})$-splitting family. We will construct a family $\{A_s : s \in 2^{<\omega}\}$ which verifies $\omega \in \mathcal{S}_{(\mathcal{J},\mathcal{I})}$(this implies that $[0,1]$ does not have $(\mathcal{I},\mathcal{J})$-BW property).

Sketch of proof

Since T is an infinite tree with finite branches, by König's lemma, there exists $b \in r^\omega$ such that $X \setminus A_{b|n} \in \mathcal{J}$ for every $n \in \omega$. According to the fact that ω is an $(\mathcal{J},\mathcal{I})$-small set we have that $X \in \mathcal{I}$. Contradiction.
$(2) \Rightarrow (3)$ Suppose that $\mathfrak{s}(\mathcal{J},\mathcal{I}) = \omega$, and $\{S_n : n \in \omega\}$ be a $(\mathcal{J},\mathcal{I})$-splitting family. We will construct a family $\{A_s : s \in 2^{<\omega}\}$ which verifies $\omega \in \mathcal{S}_{(\mathcal{J},\mathcal{I})}$(this implies that $[0,1]$ does not have $(\mathcal{I},\mathcal{J})$-BW property).

Sketch of proof

First, take $A_\emptyset = \omega$, and let $n_\emptyset$ be the smallest n such that S_n splits ω. Put

$$A_0 = A_\emptyset \cap A_{n_\emptyset};\ A_1 = A_\emptyset \setminus A_{n_\emptyset}.$$

Then $A_0 \in \mathcal{I}^+$ and $A_1 \in \mathcal{I}^+$.

Sketch of proof

First, take $A_\emptyset = \omega$, and let $n_\emptyset$ be the smallest n such that S_n splits ω. Put

$$A_0 = A_\emptyset \cap A_{n_\emptyset};\ A_1 = A_\emptyset \setminus A_{n_\emptyset}.$$

Then $A_0 \in \mathcal{I}^+$ and $A_1 \in \mathcal{I}^+$.

Sketch of proof

Suppose that we have already constructed A_s for all $s \in 2^n$. Then for each $s \in 2^n$, $A_s \in \mathcal{I}^+$. Let n_s be the smallest n such that S_n splits A_s. Put

$$A_{s^\frown 0} = A_s \cap S_{n_s},\ A_{s^\frown 1} = A_s \setminus S_{n_s}.$$

According to the definition of $(\mathcal{J},\mathcal{I})$-splitting family, both of $A_{s^\frown 0}$ and $A_{s^\frown 1}$ are in $\mathcal{I}^+$. This allows us to keep this proceed going and then we finish our construction. Clearly, the family $\{A_s : s \in 2^{<\omega}\}$ satisfies $S_1 - S_3$, it is enough to show that this family also satisfies the condition S_4. For every $b \in 2^\omega$, every $X \subset \omega$ with $X \setminus A_{b|n} \in \mathcal{J}$ for every $n \in \omega$. Suppose that $X \in \mathcal{I}^+$. Let n_X be the smallest n such that S_n splits X. Since $X \setminus A_{b|n} \in \mathcal{J}$ for every $n \in \omega$, so S_{n_X} splits $A_{b|n}$ for every $n \in \omega$. Hence, there is $k \leq n_X$ such that $S_{n_{b|k}} = S_{n_X}$. Then either $A_{b|k+1} = A_{b|k} \cap S_{n_X}$ or $A_{b|k+1} = A_{b|k} \setminus S_{n_X}$. This implies that S_{n_X} does not split $A_{b|k+1}$, which is a contradiction. Therefore, the family $\{A_s : s \in 2^{<\omega}\}$ also satisfies S_4.

Sketch of proof

Suppose that we have already constructed A_s for all $s \in 2^n$. Then for each $s \in 2^n$, $A_s \in \mathcal{I}^+$. Let n_s be the smallest n such that S_n splits A_s. Put

$$A_{s^\frown 0} = A_s \cap S_{n_s},\ A_{s^\frown 1} = A_s \setminus S_{n_s}.$$

According to the definition of $(\mathcal{J},\mathcal{I})$-splitting family, both of $A_{s^\frown 0}$ and $A_{s^\frown 1}$ are in $\mathcal{I}^+$. This allows us to keep this proceed going and then we finish our construction. Clearly, the family $\{A_s : s \in 2^{<\omega}\}$ satisfies $S_1 - S_3$, it is enough to show that this family also satisfies the condition S_4. For every $b \in 2^\omega$, every $X \subset \omega$ with $X \setminus A_{b|n} \in \mathcal{J}$ for every $n \in \omega$. Suppose that $X \in \mathcal{I}^+$. Let n_X be the smallest n such that S_n splits X. Since $X \setminus A_{b|n} \in \mathcal{J}$ for every $n \in \omega$, so S_{n_X} splits $A_{b|n}$ for every $n \in \omega$. Hence, there is $k \leq n_X$ such that $S_{n_{b|k}} = S_{n_X}$. Then either $A_{b|k+1} = A_{b|k} \cap S_{n_X}$ or $A_{b|k+1} = A_{b|k} \setminus S_{n_X}$. This implies that S_{n_X} does not split $A_{b|k+1}$, which is a contradiction. Therefore, the family $\{A_s : s \in 2^{<\omega}\}$ also satisfies S_4.

Sketch of proof

Suppose that we have already constructed A_s for all $s \in 2^n$. Then for each $s \in 2^n$, $A_s \in \mathcal{I}^+$. Let n_s be the smallest n such that S_n splits A_s. Put

$$A_{s^\frown 0} = A_s \cap S_{n_s},\ A_{s^\frown 1} = A_s \setminus S_{n_s}.$$

According to the definition of $(\mathcal{J},\mathcal{I})$-splitting family, both of $A_{s^\frown 0}$ and $A_{s^\frown 1}$ are in $\mathcal{I}^+$. This allows us to keep this proceed going and then we finish our construction. Clearly, the family $\{A_s : s \in 2^{<\omega}\}$ satisfies $S_1 - S_3$, it is enough to show that this family also satisfies the condition S_4. For every $b \in 2^\omega$, every $X \subset \omega$ with $X \setminus A_{b|n} \in \mathcal{J}$ for every $n \in \omega$. Suppose that $X \in \mathcal{I}^+$. Let n_X be the smallest n such that S_n splits X. Since $X \setminus A_{b|n} \in \mathcal{J}$ for every $n \in \omega$, so S_{n_X} splits $A_{b|n}$ for every $n \in \omega$. Hence, there is $k \leq n_X$ such that $S_{n_{b|k}} = S_{n_X}$. Then either $A_{b|k+1} = A_{b|k} \cap S_{n_X}$ or $A_{b|k+1} = A_{b|k} \setminus S_{n_X}$. This implies that S_{n_X} does not split $A_{b|k+1}$, which is a contradiction. Therefore, the family $\{A_s : s \in 2^{<\omega}\}$ also satisfies S_4.

Our results

Theorem

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, then the following conditions are equivalent:

(1) $(\mathcal{I},\mathcal{J})$ *is* $Ramsey^*$,

(2) $(\mathcal{I},\mathcal{J})$ *is* Mon^*,

(3) $[0,1]$ *has* $(\mathcal{I},\mathcal{J})$*-BW.*

Sketch of proof

(1) $\Rightarrow$ (2) Let $\langle x_n : n \in \omega \rangle$ be a sequence in $[0,1]$, define a coloring c: $[\omega]^2 \to \{0,1\}$ by

$$c(\{n,m\}) = 0 \text{ if } n < m \text{ and } x_n \leq x_m;\ c(\{n,m\}) = 1, \text{ otherwise.}$$

Since $(\mathcal{I},\mathcal{J})$ is $Ramsey^*$, there exists $A \in \mathcal{I}^+$ such that A is $\mathcal{J}$-homogeneous for c. So we may assume that for every $n \in A$,

$$\{m : c(\{n,m\}) = 1\} \in \mathcal{J}.$$

Therefore, $\langle x_n : n \in A \rangle$ is $\mathcal{J}$-increasing.

Sketch of proof

$(1) \Rightarrow (2)$ Let $\langle x_n : n \in \omega \rangle$ be a sequence in $[0,1]$, define a coloring c: $[\omega]^2 \to \{0,1\}$ by

$$c(\{n,m\}) = 0 \text{ if } n < m \text{ and } x_n \le x_m;\ c(\{n,m\}) = 1, \text{ otherwise.}$$

Since $(\mathcal{I},\mathcal{J})$ is $Ramsey^*$, there exists $A \in \mathcal{I}^+$ such that A is $\mathcal{J}$-homogeneous for c. So we may assume that for every $n \in A$,

$$\{m : c(\{n,m\}) = 1\} \in \mathcal{J}.$$

Therefore, $\langle x_n : n \in A \rangle$ is $\mathcal{J}$-increasing.

Sketch of proof

$(1) \Rightarrow (2)$ Let $\langle x_n : n \in \omega\rangle$ be a sequence in $[0,1]$, define a coloring c: $[\omega]^2 \to \{0,1\}$ by

$c(\{n,m\}) = 0$ if $n < m$ and $x_n \leq x_m$; $c(\{n,m\}) = 1$, otherwise.

Since $(\mathcal{I},\mathcal{J})$ is $Ramsey^*$, there exists $A \in \mathcal{I}^+$ such that A is $\mathcal{J}$-homogeneous for c. So we may assume that for every $n \in A$,

$$\{m : c(\{n,m\}) = 1\} \in \mathcal{J}.$$

Therefore, $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-increasing.

Sketch of proof

$(1) \Rightarrow (2)$ Let $\langle x_n : n \in \omega \rangle$ be a sequence in $[0, 1]$, define a coloring c: $[\omega]^2 \to \{0, 1\}$ by

$c(\{n, m\}) = 0$ if $n < m$ and $x_n \leq x_m$; $c(\{n, m\}) = 1$, otherwise.

Since $(\mathcal{I}, \mathcal{J})$ is $Ramsey^*$, there exists $A \in \mathcal{I}^+$ such that A is $\mathcal{J}$-homogeneous for c. So we may assume that for every $n \in A$,

$$\{m : c(\{n, m\}) = 1\} \in \mathcal{J}.$$

Therefore, $\langle x_n : n \in A \rangle$ is $\mathcal{J}$-increasing.

Sketch of proof

$(2) \Rightarrow (3)$ Assume that $(\mathcal{I},\mathcal{J})$ is Mon^*.
For a given sequence $\langle x_n : n \in \omega\rangle$ in $[0,1]$, there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-monotone.
We may assume that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-nondecreasing. Let

$$x = sup_{n\in A} x_n.$$

For any $\varepsilon > 0$, there is $x_N \in A$ such that $x_N > x - \varepsilon$. Then

$$\{n \in A : |x_n - x| \geq \varepsilon\} \subseteq \{n \in A : x_N > x_n\} \in \mathcal{J}.$$

Thus, $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-convergent to x.

Sketch of proof

$(2) \Rightarrow (3)$ Assume that $(\mathcal{I},\mathcal{J})$ is Mon^*.
For a given sequence $\langle x_n : n \in \omega\rangle$ in $[0,1]$, there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-monotone.
We may assume that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-nondecreasing. Let

$$x = sup_{n\in A} x_n.$$

For any $\varepsilon > 0$, there is $x_N \in A$ such that $x_N > x - \varepsilon$. Then

$$\{n \in A : |x_n - x| \geq \varepsilon\} \subseteq \{n \in A : x_N > x_n\} \in \mathcal{J}.$$

Thus, $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-convergent to x.

Sketch of proof

$(2) \Rightarrow (3)$ Assume that $(\mathcal{I},\mathcal{J})$ is Mon^*.
For a given sequence $\langle x_n : n \in \omega\rangle$ in $[0,1]$, there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-monotone.
We may assume that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-nondecreasing. Let

$$x = sup_{n\in A} x_n.$$

For any $\varepsilon > 0$, there is $x_N \in A$ such that $x_N > x - \varepsilon$. Then

$$\{n \in A : |x_n - x| \geq \varepsilon\} \subseteq \{n \in A : x_N > x_n\} \in \mathcal{J}.$$

Thus, $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-convergent to x.

Sketch of proof

$(2) \Rightarrow (3)$ Assume that $(\mathcal{I},\mathcal{J})$ is Mon^*.
For a given sequence $\langle x_n : n \in \omega\rangle$ in $[0,1]$, there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-monotone.
We may assume that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-nondecreasing. Let

$$x = sup_{n\in A} x_n.$$

For any $\varepsilon > 0$, there is $x_N \in A$ such that $x_N > x - \varepsilon$. Then

$$\{n \in A : |x_n - x| \geq \varepsilon\} \subseteq \{n \in A : x_N > x_n\} \in \mathcal{J}.$$

Thus, $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-convergent to x.

Sketch of proof

$(2) \Rightarrow (3)$ Assume that $(\mathcal{I},\mathcal{J})$ is Mon^*.
For a given sequence $\langle x_n : n \in \omega\rangle$ in $[0,1]$, there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-monotone.
We may assume that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-nondecreasing. Let

$$x = sup_{n\in A} x_n.$$

For any $\varepsilon > 0$, there is $x_N \in A$ such that $x_N > x - \varepsilon$. Then

$$\{n \in A : |x_n - x| \geq \varepsilon\} \subseteq \{n \in A : x_N > x_n\} \in \mathcal{J}.$$

Thus, $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-convergent to x.

Sketch of proof

$(2) \Rightarrow (3)$ Assume that $(\mathcal{I},\mathcal{J})$ is Mon^*.
For a given sequence $\langle x_n : n \in \omega \rangle$ in $[0,1]$, there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A \rangle$ is $\mathcal{J}$-monotone.
We may assume that $\langle x_n : n \in A \rangle$ is $\mathcal{J}$-nondecreasing. Let

$$x = sup_{n \in A} x_n.$$

For any $\varepsilon > 0$, there is $x_N \in A$ such that $x_N > x - \varepsilon$. Then

$$\{n \in A : |x_n - x| \geq \varepsilon\} \subseteq \{n \in A : x_N > x_n\} \in \mathcal{J}.$$

Thus, $\langle x_n : n \in A \rangle$ is $\mathcal{J}$-convergent to x.

Sketch of proof

$(3)\Rightarrow(1)$ Let $r\in\omega$, and c: $[\omega]^2\to\{0,\cdots,r-1\}$ being a coloring of $[\omega]^2$.

We shall define a family $\{A_s : s\in r^{<\omega}\}$ that satisfies S_1-S_3 as follows

- $A_\emptyset=\omega$,
- $A_{s\frown i}=\{n\in A_s : c(lh(s\frown i),n)=i\}$, $i\in\{0,\cdots,r-1\}$.

Note that $[0,1]$ has $(\mathcal{I},\mathcal{J})$-BW, so ω is not a $(\mathcal{J},\mathcal{I})$-small set, this implies that there are $x\in r^\omega$ and $B\in\mathcal{I}^+$ such that $B\setminus A_{x|n}\in\mathcal{J}$ for all $n\in\omega$. Then there exists $i\in\{0,\cdots,r-1\}$, and $C\subseteq B$ with $C\in\mathcal{I}^+$ such that $x(k-1)=i$ for every $k\in C$.
It is not hard to see that for every $n\in C$,

$$\{k\in C : c(\{n,k\})\neq i\}\subseteq C\setminus A_{x|n}\in\mathcal{J}.$$

This implies that C is $\mathcal{J}$-homogeneous as desired.

Sketch of proof

$(3) \Rightarrow (1)$ Let $r \in \omega$, and c: $[\omega]^2 \to \{0, \cdots, r-1\}$ being a coloring of $[\omega]^2$.
We shall define a family $\{A_s : s \in r^{<\omega}\}$ that satisfies S_1-S_3 as follows

- $A_\emptyset = \omega$,
- $A_{s\frown i} = \{n \in A_s : c(lh(s \frown i), n) = i\}$, $i \in \{0, \cdots, r-1\}$.

Note that $[0,1]$ has $(\mathcal{I},\mathcal{J})$-BW, so ω is not a $(\mathcal{J},\mathcal{I})$-small set, this implies that there are $x \in r^\omega$ and $B \in \mathcal{I}^+$ such that $B \setminus A_{x|n} \in \mathcal{J}$ for all $n \in \omega$. Then there exists $i \in \{0, \cdots, r-1\}$, and $C \subseteq B$ with $C \in \mathcal{I}^+$ such that $x(k-1) = i$ for every $k \in C$.
It is not hard to see that for every $n \in C$,

$$\{k \in C : c(\{n,k\}) \neq i\} \subseteq C \setminus A_{x|n} \in \mathcal{J}.$$

This implies that C is $\mathcal{J}$-homogeneous as desired.

Sketch of proof

(3) $\Rightarrow$ (1) Let $r \in \omega$, and c: $[\omega]^2 \to \{0, \cdots, r-1\}$ being a coloring of $[\omega]^2$.
We shall define a family $\{A_s : s \in r^{<\omega}\}$ that satisfies S_1-S_3 as follows

- $A_\emptyset = \omega$,
- $A_{s\frown i} = \{n \in A_s : c(lh(s \frown i), n) = i\}$, $i \in \{0, \cdots, r-1\}$.

Note that $[0,1]$ has $(\mathcal{I},\mathcal{J})$-BW, so ω is not a $(\mathcal{J},\mathcal{I})$-small set, this implies that there are $x \in r^\omega$ and $B \in \mathcal{I}^+$ such that $B \setminus A_{x|n} \in \mathcal{J}$ for all $n \in \omega$. Then there exists $i \in \{0, \cdots, r-1\}$, and $C \subseteq B$ with $C \in \mathcal{I}^+$ such that $x(k-1) = i$ for every $k \in C$.
It is not hard to see that for every $n \in C$,

$$\{k \in C : c(\{n,k\}) \neq i\} \subseteq C \setminus A_{x|n} \in \mathcal{J}.$$

This implies that C is $\mathcal{J}$-homogeneous as desired.

Sketch of proof

$(3) \Rightarrow (1)$ Let $r \in \omega$, and c: $[\omega]^2 \to \{0, \cdots, r-1\}$ being a coloring of $[\omega]^2$.
We shall define a family $\{A_s : s \in r^{<\omega}\}$ that satisfies S_1-S_3 as follows

- $A_\emptyset = \omega$,
- $A_{s \frown i} = \{n \in A_s : c(lh(s \frown i), n) = i\}$, $i \in \{0, \cdots, r-1\}$.

Note that $[0,1]$ has $(\mathcal{I},\mathcal{J})$-BW, so ω is not a $(\mathcal{J},\mathcal{I})$-small set, this implies that there are $x \in r^\omega$ and $B \in \mathcal{I}^+$ such that $B \setminus A_{x|n} \in \mathcal{J}$ for all $n \in \omega$. Then there exists $i \in \{0, \cdots, r-1\}$, and $C \subseteq B$ with $C \in \mathcal{I}^+$ such that $x(k-1) = i$ for every $k \in C$.
It is not hard to see that for every $n \in C$,

$$\{k \in C : c(\{n,k\}) \neq i\} \subseteq C \setminus A_{x|n} \in \mathcal{J}.$$

This implies that C is $\mathcal{J}$-homogeneous as desired.

Sketch of proof

$(3) \Rightarrow (1)$ Let $r \in \omega$, and $c\colon [\omega]^2 \to \{0,\cdots,r-1\}$ being a coloring of $[\omega]^2$.
We shall define a family $\{A_s : s \in r^{<\omega}\}$ that satisfies S_1-S_3 as follows

- $A_\emptyset = \omega$,
- $A_{s\frown i} = \{n \in A_s : c(lh(s \frown i), n) = i\}$, $i \in \{0,\cdots,r-1\}$.

Note that $[0,1]$ has $(\mathcal{I},\mathcal{J})$-BW, so ω is not a $(\mathcal{J},\mathcal{I})$-small set, this implies that there are $x \in r^\omega$ and $B \in \mathcal{I}^+$ such that $B \setminus A_{x|n} \in \mathcal{J}$ for all $n \in \omega$. Then there exists $i \in \{0,\cdots,r-1\}$, and $C \subseteq B$ with $C \in \mathcal{I}^+$ such that $x(k-1) = i$ for every $k \in C$.
It is not hard to see that for every $n \in C$,

$$\{k \in C : c(\{n,k\}) \neq i\} \subseteq C \setminus A_{x|n} \in \mathcal{J}.$$

This implies that C is $\mathcal{J}$-homogeneous as desired.

Our results

Theorem

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω such that $(\mathcal{I},\mathcal{J})$ is weak selective. For the following conditions:

(1) $[0,1]$ *has* $(\mathcal{I},\mathcal{J})$*-BW;*

(2) *For every $r \in \omega$, every family $\{A_s : s \in r^{<\omega}\}$ fulfilling conditions S_1-S_3, there are $x \in r^\omega$ and $C \in \mathcal{J}^+$ such that $C \subseteq^* A_{x|n}$ for each $n \in \omega\}$.*

(3) $[0,1]$ *has* $(\mathcal{J},\mathcal{I})$*-BW.*

It holds that $(1) \Rightarrow (2) \Rightarrow (3)$.

Sketch of proof

$(1) \Rightarrow (2)$ Note that $[0,1]$ has $(\mathcal{I},\mathcal{J})$-BW implies that $\omega \notin \mathcal{S}_{(\mathcal{J},\mathcal{I})}$.
So for every $r \in \omega$, every family $\{A_s : s \in r^{<\omega}\}$ fulfilling conditions S_1-S_3, there are $x \in r^\omega$ and $B \in \mathcal{I}^+$ such that $B \setminus A_{x|n} \in \mathcal{J}$ for every $n \in \omega$.
It is easy to see that

$$B \setminus A_{x|1},\ B \cap (A_{x|2} \setminus A_{x|1}),\ \cdots,\ B \cap (A_{x|n+1} \setminus A_{x|n}),\ \cdots$$

is a partition of B into sets from $\mathcal{J}$.
Note that $(\mathcal{I},\mathcal{J})$ is weak selective, so $\mathcal{J}|B$ is locally selective.
Thus, there exists $C \subset B$ with $C \in \mathcal{J}^+$ such that $|C \cap B \setminus A_{x|1}| \leq 1$, $|C \cap B \cap (A_{x|2} \setminus A_{x|n})| \leq 1$ for every $n \in \omega$. It is easy to check that the set C is desired.

Sketch of proof

$(1)\Rightarrow(2)$ Note that $[0,1]$ has $(\mathcal{I},\mathcal{J})$-BW implies that $\omega\notin\mathcal{S}_{(\mathcal{J},\mathcal{I})}$. So for every $r\in\omega$, every family $\{A_s : s\in r^{<\omega}\}$ fulfilling conditions S_1-S_3, there are $x\in r^\omega$ and $B\in\mathcal{I}^+$ such that $B\setminus A_{x|n}\in\mathcal{J}$ for every $n\in\omega$.

It is easy to see that

$$B\setminus A_{x|1},\ B\cap(A_{x|2}\setminus A_{x|1}),\ \cdots,\ B\cap(A_{x|n+1}\setminus A_{x|n}),\ \cdots$$

is a partition of B into sets from $\mathcal{J}$.

Note that $(\mathcal{I},\mathcal{J})$ is weak selective, so $\mathcal{J}|B$ is locally selective.

Thus, there exists $C\subset B$ with $C\in\mathcal{J}^+$ such that $|C\cap B\setminus A_{x|1}|\leq 1$, $|C\cap B\cap(A_{x|2}\setminus A_{x|n})|\leq 1$ for every $n\in\omega$. It is easy to check that the set C is desired.

Sketch of proof

$(1) \Rightarrow (2)$ Note that $[0,1]$ has $(\mathcal{I},\mathcal{J})$-BW implies that $\omega \notin \mathcal{S}_{(\mathcal{J},\mathcal{I})}$. So for every $r \in \omega$, every family $\{A_s : s \in r^{<\omega}\}$ fulfilling conditions S_1-S_3, there are $x \in r^\omega$ and $B \in \mathcal{I}^+$ such that $B \setminus A_{x|n} \in \mathcal{J}$ for every $n \in \omega$.
It is easy to see that

$$B \setminus A_{x|1},\ B \cap (A_{x|2} \setminus A_{x|1}),\ \cdots,\ B \cap (A_{x|n+1} \setminus A_{x|n}),\ \cdots$$

is a partition of B into sets from $\mathcal{J}$.
Note that $(\mathcal{I},\mathcal{J})$ is weak selective, so $\mathcal{J}|B$ is locally selective. Thus, there exists $C \subset B$ with $C \in \mathcal{J}^+$ such that $|C \cap B \setminus A_{x|1}| \leq 1$, $|C \cap B \cap (A_{x|2} \setminus A_{x|n})| \leq 1$ for every $n \in \omega$. It is easy to check that the set C is desired.

Sketch of proof

$(1) \Rightarrow (2)$ Note that $[0,1]$ has $(\mathcal{I},\mathcal{J})$-BW implies that $\omega \notin \mathcal{S}_{(\mathcal{J},\mathcal{I})}$. So for every $r \in \omega$, every family $\{A_s : s \in r^{<\omega}\}$ fulfilling conditions S_1-S_3, there are $x \in r^{\omega}$ and $B \in \mathcal{I}^+$ such that $B \setminus A_{x|n} \in \mathcal{J}$ for every $n \in \omega$.

It is easy to see that

$$B \setminus A_{x|1},\ B \cap (A_{x|2} \setminus A_{x|1}),\ \cdots,\ B \cap (A_{x|n+1} \setminus A_{x|n}),\ \cdots$$

is a partition of B into sets from $\mathcal{J}$.

Note that $(\mathcal{I},\mathcal{J})$ is weak selective, so $\mathcal{J}|B$ is locally selective.

Thus, there exists $C \subset B$ with $C \in \mathcal{J}^+$ such that $|C \cap B \setminus A_{x|1}| \leq 1$, $|C \cap B \cap (A_{x|2} \setminus A_{x|n})| \leq 1$ for every $n \in \omega$. It is easy to check that the set C is desired.

Sketch of proof

$(2) \Rightarrow (3)$ It is enough to show that ω is not an $(\mathcal{I},\mathcal{J})$-small set.
To this end, for every $r \in \omega$, for any family $\{A_s : s \in 2^{<\omega}\}$ satisfying S_1-S_3. By (2), there are $x \in r^\omega$ and $C \in \mathcal{J}^+$ such that for each $n \in \omega$, $C \setminus A_{x|n} \in Fin \subseteq \mathcal{I}$.

Sketch of proof

$(2) \Rightarrow (3)$ It is enough to show that ω is not an $(\mathcal{I},\mathcal{J})$-small set. To this end, for every $r \in \omega$, for any family $\{A_s : s \in 2^{<\omega}\}$ satisfying S_1-S_3. By (2), there are $x \in r^\omega$ and $C \in \mathcal{J}^+$ such that for each $n \in \omega$, $C \setminus A_{x|n} \in Fin \subseteq \mathcal{I}$.

Sketch of proof

$(2) \Rightarrow (3)$ It is enough to show that ω is not an $(\mathcal{I},\mathcal{J})$-small set. To this end, for every $r \in \omega$, for any family $\{A_s : s \in 2^{<\omega}\}$ satisfying S_1-S_3. By (2), there are $x \in r^\omega$ and $C \in \mathcal{J}^+$ such that for each $n \in \omega$, $C \setminus A_{x|n} \in Fin \subseteq \mathcal{I}$.

Our results

Definition

$\mathcal{I}$ is *h-Ramsey* (respectively, *h-$Ramsey^*$*) if for every $A \in \mathcal{I}^+$, $\mathcal{I}|A$ is Ramsey (respectively, $\mathcal{I}|A$ is $Ramsey^*$).

Theorem

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω and $\mathcal{J}$ being a weak Q-ideals such that $\mathcal{I} \leq_{RB} \mathcal{J}$,

(1) *If $\mathcal{J}$ is h-$Ramsey^*$, then $\mathcal{I}$ is h-$Ramsey^*$;*

(2) *If $\mathcal{J}$ is h-Ramsey, then $\mathcal{I}$ is h-Ramsey.*

Our results

Definition

$\mathcal{I}$ is *h-Ramsey* (respectively, *h-$Ramsey^*$*) if for every $A \in \mathcal{I}^+$, $\mathcal{I}|A$ is Ramsey (respectively, $\mathcal{I}|A$ is $Ramsey^*$).

Theorem

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω and $\mathcal{J}$ being a weak Q-ideals such that $\mathcal{I} \leq_{RB} \mathcal{J}$,

(1) *If $\mathcal{J}$ is h-$Ramsey^*$, then $\mathcal{I}$ is h-$Ramsey^*$;*

(2) *If $\mathcal{J}$ is h-Ramsey, then $\mathcal{I}$ is h-Ramsey.*

Sketch of proof

The assertion (1) follows from the following lemmata.

Lemma (Theorem 4.3,[1])

h-$Ramsey^*$ is equal to h-BW property.

Lemma (Theorem 6.2, [2])

The h-BW property is preserved under the $\leq_{RB}$-order in the realm of Q-ideals.

Sketch of proof

The assertion (1) follows from the following lemmata.

Lemma (Theorem 4.3,[1])

h-$Ramsey^*$ is equal to h-BW property.

Lemma (Theorem 6.2, [2])

The h-BW property is preserved under the $\leq_{RB}$-order in the realm of Q-ideals.

Sketch of proof

The key in the proof of the assertion (2) is the following.

Lemma (Theorem 3.16, [1])

$\mathcal{I}$ is h-Ramsey if, and only if $\mathcal{I}$ is h-Fin-BW and being a weak Q-ideal.

Claim

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, and $\mathcal{J}$ being a Q-ideal. If $\mathcal{I} \leq_{KB} \mathcal{J}$ then $\mathcal{I}$ is also a Q-ideal.

Claim

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, and $\mathcal{J}$ being a weak Q-point. If $\mathcal{I} \leq_{RB} \mathcal{J}$ then $\mathcal{I}$ is also a weak Q-ideal.

Sketch of proof

The key in the proof of the assertion (2) is the following.

Lemma (Theorem 3.16, [1])

$\mathcal{I}$ is h-Ramsey if, and only if $\mathcal{I}$ is h-Fin-BW and being a weak Q-ideal.

Claim

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, and $\mathcal{J}$ being a Q-ideal. If $\mathcal{I} \leq_{KB} \mathcal{J}$ then $\mathcal{I}$ is also a Q-ideal.

Claim

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, and $\mathcal{J}$ being a weak Q-point. If $\mathcal{I} \leq_{RB} \mathcal{J}$ then $\mathcal{I}$ is also a weak Q-ideal.

Sketch of proof

The key in the proof of the assertion (2) is the following.

Lemma (Theorem 3.16, [1])

$\mathcal{I}$ is h-Ramsey if, and only if $\mathcal{I}$ is h-Fin-BW and being a weak Q-ideal.

Claim

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, and $\mathcal{J}$ being a Q-ideal. If $\mathcal{I} \leq_{KB} \mathcal{J}$ then $\mathcal{I}$ is also a Q-ideal.

Claim

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, and $\mathcal{J}$ being a weak Q-point. If $\mathcal{I} \leq_{RB} \mathcal{J}$ then $\mathcal{I}$ is also a weak Q-ideal.

Reference

- R. Filipów, N. Mrożek, I. Recław and P. Szuca, Ideal version of Ramsey Theorem, Czech. Math, 136 (2011), 289-308.
- R. Filipów, N. Mrożek, I. Recław and P. Szuca, Ideal Convergence of Bounded Sequences, J. Symbolic Logic, 72 (2007), 501-512.

Thank you!