Weihrauch degrees of numerical problems —comparison with arithmetic—

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Weihrauch reducibility

For $f, g \in \omega^\omega$,

- Turing reducibility: $f \leq_T g \Leftrightarrow$"$f$ is computable from g".

For $A, B \subseteq \omega^\omega$,

- Muchnik reducibility: $A \leq_w B \Leftrightarrow$
  "any element $f \in B$ computes an element $f \geq_T g \in A$",
- Medvedev reducibility: $A \leq_s B \Leftrightarrow$
  "there is a uniform method Φ to convert an element $f \in B$ into an element $\Phi^f = g \in A$".

For $P, Q \subseteq \omega^\omega \times \omega^\omega$,

- Computable reducibility: $P \leq_c Q$,
- Weihrauch reducibility: $P \leq_W Q$.

Weihrauch reducibility

Consider $P \subseteq \omega^\omega \times \omega^\omega$ as $P :\subseteq \omega^\omega \to \mathcal{P}(\omega^\omega) \setminus \{\emptyset\}$.

- Computable reducibility: $P \leq_c Q \Leftrightarrow$

  $\forall f \in \mathrm{dom}(P) \exists g \leq_T f$ such that $g \in \mathrm{dom}(Q)$ and $P(f) \leq_w^f Q(g)$
  (i.e., $\forall u \in Q(g)\, \exists v \leq_T u \oplus f$ such that $u \in P(f)$)
- Weihrauch reducibility: $P \leq_W Q \Leftrightarrow$

  there are Turing functionals Φ, Ψ such that
  $\forall f \in \mathrm{dom}(P)\ \Phi^f = g \in \mathrm{dom}(Q)$ and $P(f) \leq_s Q(g)$ via Ψ^f
  (i.e., $\forall u \in Q(g)\ \Psi^{u \oplus f} = v \in P(f)$)

P describes a problem of the form $\forall f \exists g (\varphi(f) \to \psi(f, g))$.

- $\leq_W$ is often considered as a reduction on Π^1_2-problems (but not really).
- $f \in \mathrm{dom}(P)$: instance/input of a problem P.
- $g \in P(f)$: P-solution/output for g.

Weihrauch lattice

Degrees induced by Weihrauch reducibility form a lattice.

- $\sup(P, Q) = P \sqcup Q$
  $= \{((0, f), g) : (f, g) \in P\} \cup \{((1, f), g) : (f, g) \in Q\}$
- $\inf(P, Q) = P \sqcap Q$
  $= \{((f, g), (0, h)) : (f, g) \in \mathrm{dom}(P) \times \mathrm{dom}(Q), (f, h) \in P\}$
  $\cup \{((f, g), (1, h)) : (f, g) \in \mathrm{dom}(P) \times \mathrm{dom}(Q), (g, h) \in Q\}$
- $\mathbf{0}$: a problem with empty domain (i.e., $\mathbf{0} = \emptyset$): easiest problem

* One may add ∞ as the hardest problem: $\mathrm{dom}(\infty) = \omega^\omega$, $\infty(f) = \emptyset$

Here, we mainly focus on problems harder than "self-solvable".

- $\mathbf{1} := \mathrm{id} = \{(f, f) : f \in \omega^\omega\}$: self-solvable (trivial) problem

Product is a basic operator on the Weihrauch lattice.

- $P \times Q = \{((f, g), (u, v)) : (f, u) \in P, (g, v) \in Q\}$
  ($P \times Q \geq_W \sup(P, Q)$ if $P, Q \geq_W \mathrm{id}$.)

Weihrauch degrees I

$\mathcal{X}$: Polish space with computable representation

- $\mathsf{C}_{\mathcal{X}}$ (closed choice on $\mathcal{X}$)

  instance: (a negative code for) a closed set $A \subseteq \mathcal{X}$
  solution: a point in A
- $\mathsf{K}_{\mathcal{X}}$ (compact choice on $\mathcal{X}$)

  instance: (a code by 2^{-n}-nets for) a compact set $A \subseteq \mathcal{X}$
  solution: a point in A
- $\lim_{\mathcal{X}}$ (limit operator)

  instance: a convergent sequence $\langle x_i \rangle_{i \in \omega}$
  solution: $x = \lim x_i$
- $\mathsf{BWT}_{\mathcal{X}}$ (Bolzano-Weierstraß theorem)

  instance: totally bounded sequence $\langle x_i \rangle_{i \in \omega}$
  solution: convergent subsequence of $\langle x_i \rangle_{i \in \omega}$
- IVT (intermediate value theorem)

  instance: continuous function $f : [0,1] \to \mathbb{R}$ such that $f(0)f(1) \leq 0$
  solution: $x \in [0,1]$ such that $f(x) = 0$

Weihrauch degrees II

- WKL (weak König's lemma)

  instance: infinite tree $T \subseteq 2^{<\omega}$
  solution: a path of T
- WWKL (weak weak König's lemma)

  instance: infinite tree $T \subseteq 2^{<\omega}$ with positive measure
  solution: a path of T
- MLR (Martin-Löf random)

  instance: $x \in \mathbb{R}$
  solution: Martin-Löf random real relative to x
- RT^n_k (Ramsey's theorem)

  instance: function $f : [\mathbb{N}]^n \to k$
  solution: an infinite homogeneous set for f
- $\mathrm{RT}^n_{<\infty}$ (Ramsey's theorem)

  instance: $k \in \omega$ and function $f : [\mathbb{N}]^n \to k$
  solution: an infinite homogeneous set for f

$\vdots$

Zoo of Weihrauch degrees

- There are so many results on the study of the structure of Weihrauch degrees.

  Brattka, Pauly, Marcone, Dzhafarov,...

Zoo from V. Brattka's Tutorial slides.
See http://cca-net.de/publications/weibib.php.

Too complicated???
⇒ want some reasonable measure for Weihrauch degrees.

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Two viewpoints for axioms of second-order arithmetic

A, B axioms of second-order arithmetic (including RCA_0).

Degree-theoretic strength:

- Consider the complexity of $S \subseteq \mathcal{P}(\omega)$ such that $(\omega, S) \models A$.
- Strength can be described as the complexity of Turing ideals.
- Observation (though not exactly accurate)
  “$(\omega, S) \models A \Rightarrow (\omega, S) \models B$ for any S means A plus strong enough induction implies B.”

First-order strength/proof-theoretic strength

- Consider the class of first-order/Π^1_1-consequences of A.
- It can be compared with the hierarchy of induction/bounding principles.

Two viewpoints for Weihrauch degrees?

Degree-theoretic strength:

- Computable reduction $\leq_c$ well reflects Turing-degree-theoretic strength.
- Turing-degree-theoretic part of P:

  $$^{Td}(P) := \{(f,g) \in \omega^\omega : f = f_0, g \geq_T g_0 \text{ for some } (f_0,g_0) \in P\}.$$

  Then, $^{Td}(P) \leq_W P$ and $Q \leq_c P \Rightarrow Q \leq_c {}^{Td}(P)$.

First-order strength?

- Is there a good measure corresponding to the first-order parts in arithmetic?

Numerical/first-order problems

(Identify $n \in \omega$ with the constant function $\lambda x.n \in \omega^\omega$.)

- A problem P is said to be numerical/first-order if $P(f) \subseteq \omega$ for any $f \in \mathrm{dom}(P)$.

* Note that any solution of P doesn't have any computational power since it is just a constant function.

- There are many non-trivial first-order problems, e.g., $\mathsf{C}_2, \mathsf{C}_{\mathbb{N}}, \lim_{\mathbb{N}}, \dots$

Theorem (Numerical/first-order part)

For a given problem P, the numerical/first-order part of P

$$^1(P) := \max\{Q \leq_W P : Q \text{ is first-order}\}$$

always exists.

- Then, $^1(P) \leq_W P$, and, $Q \leq_W P \Rightarrow Q \leq_W {}^1(P)$ for any numerical Q.

Numerical/first-order parts

The first-order part just describes "non-uniformity" of a problem.

Theorem

A problem P is computably trivial (i.e., $P \leq_c \mathrm{id}$) if and only if $P \leq_W Q$ for some first-order problem Q.

Indeed, it is orthogonal to the degree theoretic part.

Theorem

Let $P \geq_W \mathrm{id}$.

1. ${}^{Td}({}^{Td}(P)) = {}^{Td}(P)$ *and* ${}^{1}({}^{1}(P)) = {}^{1}(P)$.
2. ${}^{Td}({}^{1}(P)) \equiv_W {}^{1}({}^{Td}(P)) \equiv_W \mathrm{id}$.

Numerical/first-order parts

Note that $^{Td}(P)$ and $^{1}(P)$ do not capture the exact power of P.

- Let $P = \inf(\mathrm{WKL}, \mathrm{C}_{\mathbb{N}})$. Then, $^{Td}(P) \equiv_W {}^{1}(P) \equiv_W \mathrm{id}$, but $P >_W \mathrm{id}$.

* Similar problem happens in arithmetic, e.g., $\mathrm{WKL} \vee \mathrm{I}\Sigma^0_2$ implies neither the existence of non-recursive set nor non-trivial induction over RCA_0.

The notion of non-diagonalizability introduced by Hirschfeld and Jockusch provides a nice condition to be first-order trivial.

Theorem (nondiagonalizable vs first-order trivial)

If a problem P is non-diagonalizable, i.e., there is a Turing functional Ψ such that

$$\Psi^f(\sigma) = 0 \Leftrightarrow \exists g \supseteq \sigma (g \in P(f)) \text{ for any } f \in \mathrm{dom}(P),$$

then, $^{1}(P)$ is trivial.

Classification by first-order strength

Here, $P' = P \circ \lim_{\mathbb{N}^\mathbb{N}}$ (the jump of P).

- $\mathrm{id} \equiv_W {}^1(\mathrm{MLR})$

  $(\mathrm{MLR} >_W \mathrm{id})$
- $\mathrm{K}_\mathbb{N} \equiv_W {}^1(\mathrm{K}_{\mathbb{R}^n}) \equiv_W {}^1(\mathrm{WKL}) \equiv_W {}^1(\mathrm{WWKL}) \equiv_W {}^1(\mathrm{IVT})$

  $(\mathrm{K}_{\mathbb{R}^n} \geq_W \mathrm{WKL} >_W \mathrm{IVT} >_W \mathrm{K}_\mathbb{N})$
- $\mathrm{C}_\mathbb{N} \equiv {}^1(\lim_{\mathbb{N}^\mathbb{N}}) \equiv_W {}^1(\mathrm{C}_{\mathbb{R}^n}) \equiv_W {}^1(\mathrm{BWT}_{\mathbb{R}^n}) \equiv_W {}^1(\lim_\mathbb{N})$

  $(\lim_{\mathbb{N}^\mathbb{N}} \geq_W \mathrm{C}_{\mathbb{R}^n} \geq_W \mathrm{BWT}_{\mathbb{R}^n} \geq_W \lim_\mathbb{N})$
- $(\mathrm{K}_\mathbb{N})' \equiv_W \mathrm{RT}^1_{<\infty} \equiv_W {}^1((\mathrm{WKL})')$
- $(\mathrm{C}_2)^{(n)} \leq_W {}^1(\mathrm{RT}^n_2) \leq_W (\mathrm{K}_\mathbb{N})^{(n)}$

  $\vdots$

Question

Brattka's observation:

$$\mathrm{K}_\mathbb{N} <_W \mathrm{C}_\mathbb{N} <_W \mathrm{K}'_\mathbb{N} <_W \mathrm{C}'_\mathbb{N} <_W \mathrm{K}''_\mathbb{N} <_W \mathrm{C}''_\mathbb{N} <_W \cdots$$

does this hierarchy correspond to the Kirby-Paris hierarchy of induction and bounding in arithmetic?

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Problems from arithmetic I

We introduce problems corresponding to

- bounded comprehension (2nd-order form of induction),
- bounded separation (2nd-order form of bounding).

Let $\Gamma = \Sigma^0_n$ or Π^0_n.

1. Γ-truth
   - instance: $\langle A, \varphi \rangle$ where $A \subseteq \omega$ and $\varphi(X) \in \Gamma^X$,
   - solution: $i \in \{0, 1\}$ answering whether $\omega \models \varphi(A)$ or not.
2. Γ-choice
   - instance: $\langle A, \varphi_0, \varphi_1 \rangle$ where $A \subseteq \omega$ and $\varphi_i(X) \in \Gamma^X$ such that $\omega \models \varphi_0(A) \vee \varphi_1(A)$,
   - solution: $i \in \{0, 1\}$ such that $\omega \models \varphi_i(A)$.

Problems from arithmetic II

For $n \geq 1$, we may easily see that

$$\Sigma^0_n\text{-choice} \leq_W \Pi^0_n\text{-choice} \leq_W \Sigma^0_n\text{-truth} \leq_W \Sigma^0_{n+1}\text{-choice}.$$

We see later that this is strict in a strong sense.

Proposition

1. Σ^0_0-truth $\equiv_W$ Σ^0_1-choice $\equiv_W$ id.
2. *For* $n \geq 1$, Π^0_n-choice $\equiv_W$ $\mathsf{C}_2^{(n-1)} \equiv_W \mathsf{LLPO}^{(n-1)}$.
3. *For* $n \geq 1$, Σ^0_n-truth $\equiv_W$ $\mathsf{LPO}^{(n-1)}$.
4. *For* $n \geq 2$, Σ^0_n-choice $\equiv_W$ Δ^0_n-truth $\equiv_W$ $\lim_2^{(n-2)}$.

Hierarchy of problems from arithmetic

- Given a problem P, P^* is defined as follows:
  - instance: $k \in \omega$ and $\langle f_i \in \mathrm{dom}(P) : i < k \rangle$,
  - solution: $\langle g_i : i < k \rangle$ such that $g_i \in P(f_i)$.

Theorem (arithmetical hierarchy of bounded principles)

For $n \geq 1$ we have the following.

1. $(\Sigma^0_n\text{-choice})^* \not\geq_W \Pi^0_n$-choice.
2. $(\Pi^0_n\text{-choice})^* \not\geq_W \Sigma^0_n$-truth.
3. $(\Sigma^0_n\text{-truth})^* \not\geq_W \Sigma^0_{n+1}$-choice.

Thus, we have the following hierarchy for $n \geq 1$:

$$(\Sigma^0_n\text{-choice})^* <_W (\Pi^0_n\text{-choice})^* <_W (\Sigma^0_n\text{-truth})^* <_W (\Sigma^0_{n+1}\text{-choice})^*.$$

Bounded comprehension, bounded separation, least number principle

1. Γ-bC (bounded choice)
   - instance: $\langle A, \varphi, k\rangle$ where $A \subseteq \omega$, $\varphi(X, x) \in \Gamma^X$ and $k \in \omega$ such that $\omega \models \exists x < k\, \varphi(A, x)$,
   - solution: $i \in \{0, \ldots, k-1\}$ such that $\omega \models \varphi(A, i)$.
2. Γ-bLC (bounded least choice)
   - instance: $\langle A, \varphi, k\rangle$ where $A \subseteq \omega$, $\varphi(X, x) \in \Gamma^X$ and $k \in \omega$ such that $\omega \models \exists x < k\, \varphi(A, x)$,
   - solution: least $i \in \{0, \ldots, k-1\}$ such that $\omega \models \varphi(A, i)$.

Proposition

Let $n \geq 1$.

1. $(\Sigma^0_n\text{-choice})^* \equiv_W \Sigma^0_n\text{-bC} \equiv_W \Delta^0_n\text{-bLC}$.

   (corresponds to bound-Δ^0_n-CA, $\mathrm{L}\Delta^0_n$) $\approx \mathrm{I}\Delta^0_n$
2. $(\Pi^0_n\text{-choice})^* \equiv_W \Pi^0_n\text{-bC}$.

   (corresponds to bound-Σ^0_n-SEP) $\approx \mathrm{B}\Sigma^0_n$
3. $(\Sigma^0_n\text{-truth})^* \equiv_W \Sigma^0_n\text{-bLC} \equiv_W \Pi^0_n\text{-bLC}$.

   (corresponds to bound-Σ^0_n-CA, $\mathrm{L}\Sigma^0_n$) $\approx \mathrm{I}\Sigma^0_n$

Bounded problems

- A first-order problem P is said to be bounded if there is a Turing functional τ such that for any $X \in \mathrm{dom}(\mathsf{P})$ of P, $\tau^X(0)\downarrow$ and $\mathsf{P}(X) \subseteq [0, \tau^X(0)]$.
- A first-order problem P is said to be k-bounded if $\mathsf{P}(X) \subseteq [0, k]$ for any $X \in \mathrm{dom}(\mathsf{P})$.

Theorem

1. *If a problem* P *is* k*-bounded, then* C_{k+1} *is not Weihrauch reducible to* P*.*
2. *If a problem* P *is bounded, then* $\mathsf{C}_{\mathbb{N}}$ *is not Weihrauch reducible to* P*.*

Bounded part

One can consider the bounded part of a degree as well.

Theorem (Bounded part)

For a given problem P, the bounded part of P

$$^{b1}(P) := \max\{Q \leq_W P : Q \text{ is bounded}\}$$

always exists.

Here are some examples.

Theorem

- *For $n \geq 1$, $^{b1}(\lim_{\mathbb{N}^{\mathbb{N}}}^{(n-1)}) \equiv_W {}^{b1}(\mathsf{C}_{\mathbb{N}}^{(n-1)}) \equiv_W (\Sigma^0_{n+1}\text{-choice})^*$.*
- *For $n \geq 0$, $^{b1}(\mathsf{WKL}^{(n)}) \equiv_W {}^{b1}(\mathsf{K}_{\mathbb{N}}^{(n)}) \equiv_W (\Pi^0_{n+1}\text{-choice})^*$.*

Note that

$$\mathsf{WKL} <_W \lim_{\mathbb{N}^{\mathbb{N}}} <_W \mathsf{WKL}' <_W \lim'_{\mathbb{N}^{\mathbb{N}}} <_W \mathsf{WKL}'' <_W \dots$$

Question

Brattka's observation:

$$\mathsf{K}_{\mathbb{N}} <_W \mathsf{C}_{\mathbb{N}} <_W \mathsf{K}'_{\mathbb{N}} <_W \mathsf{C}'_{\mathbb{N}} <_W \mathsf{K}''_{\mathbb{N}} <_W \mathsf{C}''_{\mathbb{N}} <_W \cdots$$

Does this hierarchy correspond to the following Kirby-Paris hierarchy?

$$\mathrm{B}\Sigma_1 < \mathrm{I}\Sigma_1 < \mathrm{B}\Sigma_2 < \mathrm{I}\Sigma_2 < \mathrm{B}\Sigma_3 < \cdots$$

It seems this hierarchy reasonably fits with the hierarchy in arithmetic.

- ${}^{b1}(\mathsf{K}_{\mathbb{N}}^{(n)}) \equiv_W (\Pi^0_{n+1}\text{-choice})^*$,
- ${}^{b1}(\mathsf{C}_{\mathbb{N}}^{(n)}) \equiv_W (\Sigma^0_{n+2}\text{-choice})^*$.

However, they both closer to $\mathrm{B}\Sigma^0_n \ldots$, indeed it fits better with

$$\mathrm{B}\Sigma_1 < \mathrm{I}\Delta_2 \leq \mathrm{B}\Sigma_2 < \mathrm{I}\Delta_3 \leq \mathrm{B}\Sigma_3 < \cdots$$

Classification by bounded parts

Here are more examples:

- $(\Sigma^0_1\text{-choice})^* \equiv_W \mathrm{id} \equiv_W {}^{b1}(\mathrm{MLR})$
- $(\Pi^0_1\text{-choice})^* \equiv_W (\mathrm{C}_2)^* \equiv_W {}^{b1}(\mathrm{K}_{\mathbb{R}^n}) \equiv_W {}^{b1}(\mathrm{WKL}) \equiv_W {}^{b1}(\mathrm{IVT})$
- $(\Sigma^0_2\text{-choice})^* \equiv_W (\lim_2)^* \equiv_W {}^{b1}(\lim_{\mathbb{N}^\mathbb{N}}) \equiv_W {}^{b1}(\mathrm{C}_{\mathbb{R}^n})$
  $\equiv_W {}^{b1}(\mathrm{BWT}_{\mathbb{R}^n}) \equiv_W {}^{b1}(\lim_{\mathbb{N}})$
- $(\Pi^0_{n+1}\text{-choice})^* \equiv_W {}^{b1}(\mathrm{RT}^n_{<\infty})$

“$\mathrm{RT}^n_{<\infty}$ is conservative over $(\Pi^0_{n+1}\text{-choice})^*$ for bounded principles.”

Better understanding of Weihrauch separation

One may understand some separations in a better way:

Ex. 1: MLR $<_W$ WWKL $<_W$ WKL

$^{Td}(\text{MLR}) \equiv {}^{Td}(\text{WWKL})$, but $^{b1}(\text{MLR}) < {}^{b1}(\text{WWKL})$,

$^{b1}(\text{WWKL}) \equiv {}^{b1}(\text{WKL})$, but $^{Td}(\text{WWKL}) < {}^{Td}(\text{WKL})$.

Ex. 2: IVT $<_W$ WKL $<_W$ $C_{\mathbb{R}}$

$^{b1}(\text{IVT}) \equiv {}^{b1}(\text{WKL})$, but $^{Td}(\text{IVT}) < {}^{Td}(\text{WKL})$,

$^{Td}(\text{WKL}) \equiv {}^{Td}(C_{\mathbb{R}})$, but $^{b1}(\text{WKL}) < {}^{b1}(C_{\mathbb{R}})$.

Classification by computability strength

$$
\begin{array}{lcc}
\mathrm{id} & \equiv_c & \mathrm{IVT}, \mathrm{C}_{\mathbb{N}}, \mathrm{RT}^1 \\
\quad\wedge_c & & \\
\mathrm{WWKL} & \equiv_c & \mathrm{MLR} \\
\quad\wedge_c & & \\
\mathrm{WKL} & \equiv_c & \mathrm{C}_{\mathbb{R}}, \mathrm{C}_{2^{\mathbb{N}}}, \mathrm{BWT}_{\mathbb{R}^n} \\
\quad\wedge_c & & \\
\lim & \equiv_c & \lim_{\mathbb{R}} \\
\quad\wedge_c & & \\
\mathrm{WKL}' & \geq_c & \mathrm{RT}^2 \\
\quad\wedge_c & & \\
\lim' & & \\
\quad\wedge_c & & \\
\quad\vdots & & \\
\quad\wedge_c & & \\
\Delta^1_1\mathrm{CA} & \equiv_c & \mathrm{ATR}_1 \\
\quad\wedge_c & & \\
\mathrm{C}_{\mathbb{N}^{\mathbb{N}}} & \equiv_c & \Sigma^1_1\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}
\end{array}
$$

Classification by bounded strength

(inc. recent results with Patey and Angles D'Auriac)

$$
\begin{array}{lll}
\mathrm{id} & \equiv_{W,b1} & \mathrm{MLR}, \mathrm{DNR}, \mathrm{PA} \\
\quad \wedge_{W,b1} & & \\
\mathrm{C}_2{}^* & \equiv_{W,b1} & \mathrm{LLPO}^*, \mathrm{WKL}, \mathrm{WWKL}, \mathrm{IVT}, \mathrm{C}_{2^{\mathbb{N}}} \\
\quad \wedge_{W,b1} & & \\
\mathrm{LPO}^* & \equiv_{W,b1} & \min_{\mathbb{N}\to\mathbb{N}} \\
\quad \wedge_{W,b1} & & \\
\lim_2{}^* & \equiv_{W,b1} & \lim, \mathrm{BWT}_{\mathbb{R}^n}, \lim_{\mathbb{N}}, \mathrm{C}_{\mathbb{R}} \\
\quad \wedge_{W,b1} & & \\
\mathrm{C}_2'{}^* & \equiv_{W,b1} & \mathrm{WKL}', \mathrm{RT}^1 \\
\quad \wedge_{W,b1} & & \\
\lim_2'{}^* & \equiv_{W,b1} & \lim \star \lim \\
\quad \wedge_{W,b1} & & \\
\mathrm{C}_2''{}^* & \equiv_{W,b1} & \mathrm{WKL}'', \mathrm{RT}^2 \\
\quad \wedge_{W,b1} & & \\
\Delta^1_1\mathrm{C}_2{}^* & \equiv_{W,b1} & \Delta^1_1\mathrm{CA}, \mathrm{ATR}_1 \\
\quad \wedge_{W,b1} & & \\
\Sigma^1_1\mathrm{C}_2{}^* & \equiv_{W,b1} & \Sigma^1_1\mathrm{C}_{2^{\mathbb{N}}}, \mathrm{C}_{\mathbb{N}^{\mathbb{N}}}, \Sigma^1_1\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}
\end{array}
$$

Some questions

Question

Is there a nice characterization of a problem whose first-order part is trivial, i.e., $^1(P) \equiv_W (\mathrm{id})$?

If a problem P is non-diagonalizable, i.e., there is a Turing functional Ψ such that

$$\Psi^f(\sigma) = 0 \Leftrightarrow \exists g \supseteq \sigma (g \in P(f)) \text{ for any } f \in \mathrm{dom}(P),$$

then, $^1(P)$ is trivial.

However,

- TS^1_3 (thin set theorem for 3-colors) is not below any non-diagonalizable degree, but $^1(\mathrm{TS}^1_3)$ is trivial.

Question

What is the first-order/bounded part of RT^n_2?

Indeed, the strength of Ramsey's theorem in Weihrauch degrees is still complicated with this viewpoint.

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