Topological basis problem and $\mathbb{P}_{\max}$

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Metric spaces

A metric d on a set X is a function $d : X \times X \to [0, \infty)$ such that

- $d(x, y) = 0$ iff $x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$.

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Every metric space is T_6: $F = \bigcap\{B(F, 2^{-n}) : n \in \mathbb{N}\}$.

Metrization

Is every T_6 space metrizable?

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Example ($\mathbb{S}$)

Sorgenfrey line: $(\mathbb{R}, \langle [a, b) : a, b \in \mathbb{R} \rangle)$.

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Question

Is it true that a compact T_6 space is metrizable iff it contains no Sorgenfrey subsets?

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Question (PFA)

Is it true that every uncountable T_3 space contains an uncountable subspace of $\mathbb{R}$, $\mathbb{S}$, or $\mathbb{D}$?

Topological basis problem

For regular uncountable spaces, is there a finite collection $\mathcal{B}$ such that every other regular uncountable space X contains a subspace homeomorphic to one space from $\mathcal{B}$?

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To answer this question we are willing to use standard forcing axioms (MA, PFA,...), and/or restrict ourselves to some appropriate subclass of well-behaved spaces.

The real line and the Sorgenfrey line

Theorem (Baumgartner 1973)

PFA implies that every set of reals of cardinality $\aleph_1$ *embeds homomorphically into any uncountable separable metric space and that*

every subset of the Sorgenfrey line $(\mathbb{R}, \rightarrow)$ *of cardinality* $\aleph_1$ *embeds homomorphically into any uncountable subspace of* $(\mathbb{R}, \rightarrow)$*.*

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Note that the list $\mathcal{B}$ must have at least three elements.

HS and HL

Hereditary Lindelöfness and hereditary separability play important roles in the basis problem.

A regular space is Lindelöf if every open cover has a countable subcover.

An S space is a regular hereditarily separable (HS) space which is not Lindelöf.

An L space is a regular hereditarily Lindelöf (HL) space which is not separable.

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Fact

- *HL implies T_6.*
- *For compact/Lindelöf spaces, T_6 implies HL.*

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Theorem (Moore, 2005)

There is an L space.

L groups

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Theorem (P.-Wu,2014)

For any $n < \omega$, *there is an L group* G *such that* G^n *is an L group.*

Inner topology

For a topological space (X, τ) and a collection $\mathcal{C} \subset P(X)$, the inner topology $(X, \tau^{I,\mathcal{C}})$ induced by $\mathcal{C}$ is the topology with base $\{\{x\} \cup O^{I,\mathcal{C}} : x \in O, O \text{ is open}\}$ where $O^{I,\mathcal{C}} = \bigcup\{C \in \mathcal{C} : C \subset O\}$.

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X has HL inner topology for some countable $\mathcal{C}$ if for any open set O, $O \setminus \{C \in \mathcal{C} : C \subset O\}$ is at most countable.

Theorem (P-Todorcevic)

Assume PFA. If (X, τ) is regular and $(X, \tau^{I,\mathcal{C}})$ is HL for some countable $\mathcal{C}$, then (X, τ) either is a continuous image of a separable metric space or contains an uncountable Sorgenfrey subset.

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Theorem

In Woodin's $\mathbb{P}_{\max}$ extension $L(\mathbb{R})^{\mathbb{P}_{\max}}$, if (X, τ) is regular and $(X, \tau^{I,\mathcal{C}})$ is HL for some countable $\mathcal{C}$, then (X, τ) either is a continuous image of a separable metric space or contains an uncountable Sorgenfrey subset.

Another reason of using $\mathbb{P}_{\max}$–linear orders

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Is there an HL space that combines the topology with some Aronszajn order?

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Theorem

In P-Wu's $\mathbb{P}_{\max}$ variation which forces the basis of linear orders to be $2^n + 3$, if (X, τ) is regular and $(X, \tau^{I,\mathcal{C}})$ is HL for some countable $\mathcal{C}$, then (X, τ) either is a continuous image of a separable metric space or contains an uncountable Sorgenfrey subset.

Thank you!