Is an alternative foundation of set theory possible?

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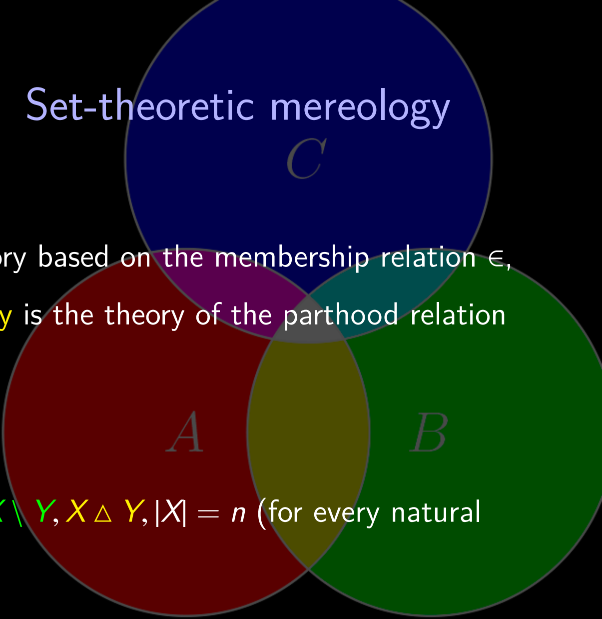
- Set theory is widely accepted as the foundation of mathematics
- The structure of set-theoretical universe is denoted by $(V, \in)$
- The 1st order language of set theory is simply $\{\in\}$

Set-theoretic mereology



In contrast to set theory based on the membership relation $\in$, set-theoretic mereology is the theory of the parthood relation $\subset$ between sets.

- Venn diagram
- $\emptyset$, $X \cup Y$, $X \cap Y$, $X \setminus Y$, $X \triangle Y$, $|X| = n$ (for every natural number n), ...

Question (Hamkins and Kikuchi 2016)

- Is parthood relation $\in$-complete in the sense that $\in$ is definable in its reduct $(V, \subset)$?
- Can set-theoretic mereology serve as a foundation of mathematics?

Set-theoretic mereology

Observation (Hamkins and Kikuchi 2016)

$(V, \subset)$ is not $\in$-complete

Theorem (Hamkins and Kikuchi 2016)

Set-theoretic mereology, namely the theory of $(V, \subset)$ is precisely the theory of an atomic unbounded relatively complemented distributive lattice. This finitely axiomatizable theory is complete and decidable.

Set-theoretic mereology

Mathematical thinking is, and must remain, essentially creative.

Emil L. Post 1944

Gödel's Thesis

A decidable complete theory can not serve as a foundation of mathematics.

Definition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures (not necessarily for the same first-order language). $\mathfrak{A}$ is a reduct of $\mathfrak{B}$ if its domain , relations and functions are definable in $\mathfrak{B}$.

Clearly, the theory of a reduct is interpretable in the theory of the original structure.

Question

- Is there any other reducts of $(V, \in)$ can serve as a foundation?
- From a Platonistic view, $(V, \in)$ might be a reduct of a finer structure of the universe of sets. What makes $(V, \in)$ so special?

Question

- Is there any other reducts of $(V, \in)$ can serve as a foundation?
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Expanding the subset relation

Observation (Hamkins and Kikuchi 2016)

$(V, \subset, \{x\})$ is $\in$-complete

$$a \in b \iff \{a\} \subset b$$

Expanding the subset relation

Definition

- $a \subset^* b$ iff $a \setminus b$ is finite
- $|a| = \infty$ iff there is a surjection from a proper subset of a to a itself

Observation

The theory of $(V, \subset, \subset^*)$ is mutually interpretable with the theory of $(V, \subset, |x| = \infty)$. The latter has a complete decidable theory.

Expanding the subset relation

We add the the unary union $\bigcup$ or the unary intersection $\bigcap$ instead.

Observation

Both $(V, \subset, \bigcup)$ and $(V, \subset, \bigcap)$ are $\in$-complete

$$y = \{x\} \Leftrightarrow \bigcup y = x \wedge |y| = 1$$
$$\Leftrightarrow \bigcap y = x \wedge |y| = 1$$

Expanding the subset relation

Observation

$(V, \cap, \bigcap)$, $(V, \cup, \bigcup)$, and $(V, \bigcap, \bigcup)$ are all $\in$-complete

$$x \subset y \Leftrightarrow \exists z \left(\bigcap z = x \wedge \bigcup z = y\right)$$

Expanding the subset relation

The power set operation P is "coarser" than the unary union

Observation

$(V, \subset, P)$ is $\in$-comlete

We can define $y = \{x\}$ as follow: Define z to be the $\subset$-least such that

$$\forall w \left((w \subset x \wedge w \neq x) \rightarrow P(w) \subset z \right)$$

Then $z = P(x) \setminus \{x\}$, so $y = P(x) \setminus z$

Question

What about the unary intersection $\bigcap$, the unary union $\bigcup$, or the power set operation P on themselves?

The unary intersection structure

The reduct $(V, \bigcap)$ is just a proper-class-branching "tree" of height $\aleph_0$ if $(V, \in)$ is well-founded

The unary union structure

Theorem (Hamkins and Y. 2017)

There is a computable complete axiomatization of the theory of $(V, \bigcup)$.

The unary union structure

There are exactly two covers of $\emptyset$:

$$\bigcup \emptyset = \emptyset$$

$$\bigcup \{\emptyset\} = \emptyset$$

Note that $\emptyset$ is the only finite set x such that $\bigcup x = x$.

The unary union structure

There are exactly two covers of $\{\emptyset\}$:

$$\bigcup\{\{\emptyset\}\} = \{\emptyset\}$$

$$\bigcup\{\emptyset, \{\emptyset\}\} = \{\emptyset\}$$

Note that it is the case for any singleton $\{a\}$.

The unary union structure

There are exactly two covers of $\{a\}$:

$$\bigcup\{\{a\}\} = \{a\}$$

$$\bigcup\{\emptyset, \{a\}\} = \{a\}$$

Note that it is the case for any singleton $\{a\}$.

The unary union structure

How many covers are there for a set of size two $\{a, b\}$?

$\{\{a, b\}\}$; $\{\emptyset, \{a, b\}\}$; $\{\{a\}, \{a, b\}\}$; $\{\{b\}, \{a, b\}\}$; $\{\{a\}, \{b\}\}$;

$\{\emptyset, \{a\}, \{a, b\}\}$; $\{\emptyset, \{b\}, \{a, b\}\}$, $\{\emptyset, \{a\}, \{b\}\}$; $\{\{a\}, \{b\}, \{a, b\}\}$;

$\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

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The unary union structure

How many covers are there for a set of size three $\{a, b, c\}$?

There are 218 covers of a set containing three elements!

- The number of covers of 4 is 64594.
- The number of covers of 5 is 4294642034.

......

The unary union structure

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The unary union structure

Fact

There is a recursive function C, given a finite set $A = \{a_1, \ldots a_n\}$, there are exactly $C(n)$ many covers of A.

$$C(n) = 2^{2^n} - \Big(\sum_{i=0}^{n-1} \binom{n}{i} \cdot C(i) \Big).$$

$$C(n) = \sum_{i=0}^{n} (-1)^{n-i} \cdot \binom{n}{i} \cdot 2^{2^i}.$$

The unary union structure

Fact

There is a recursive binary function c such that, given a finite set A of size n, there are $c(n,m)$ many covers of A of size m.

$$c(n,m) = \binom{2^n}{m} - \Big(\sum_{i=0}^{n-1} \binom{n}{i} \cdot c(i,m)\Big).$$

$$c(n,m) = \sum_{i=0}^{n} (-1)^{n-i} \cdot \binom{n}{i} \cdot \binom{2^i}{m}.$$

The infinite case

Let κ be infinite, $\lambda \geq 2$. There are $C(\kappa) = 2^{2^\kappa}$ many covers of κ, among them, there are $c(\kappa, \lambda) = [2^\kappa]^\lambda$ many covers of cardinality λ.

The unary union structure

Some definable sets in $(V, \bigcup)$:

- $x = \emptyset \Leftrightarrow \bigcup x = x \wedge \exists^{=2} y \bigcup y = x$
- $x = \{\emptyset\} \Leftrightarrow \bigcup x = \emptyset \wedge x \neq \emptyset$
- $x = \{\{\emptyset\}\} \Leftrightarrow \bigcup x = \{\emptyset\} \wedge \exists^{=2} y \bigcup y = x$
- $x = \{\emptyset, \{\emptyset\}\} \Leftrightarrow \bigcup x = \{\emptyset\} \wedge \exists^{=10} y \bigcup y = x$

The unary union structure

Observation

$\text{rank}(\bigcup x) \leq \text{rank}\, x$ for all x. In particular,

- if $\text{rank}\, x = \alpha + 1$, then $\text{rank}(\bigcup x) = \alpha$;
- else if $\text{rank}\, x = \alpha$ is limit, then $\text{rank}(\bigcup x) = \alpha$.

The unary union structure

Observation

For each $1 \leq k < \omega$ there are infinitely many sets $x \in V_{\omega+1}$ such that x have exactly k many $\bigcup$-successors in $V_{\omega+1}$ (it follows $\bigcup^k x = x$) constituting a *k*-loop.

Example

- $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}$
- $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}, \emptyset\}, \{\{\{\emptyset\}, \emptyset\}\}, \ldots\}$
- $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\{\emptyset\}\}, \emptyset\}\}, \ldots\}$

Axioms of union

Definition

D0 $|x| = 0 \leftrightarrow \bigcup x = x \wedge \exists^{=2} y \bigcup y = x$

Dn $|x| = n \leftrightarrow \bigcup x \neq x \wedge \exists^{=C(n)} y \bigcup y = x$ (for $n \geq 1$)

E1 $|x| \geq 0 \leftrightarrow x = x$

En $|x| \geq n \leftrightarrow \exists^{\geq C(n)} y \bigcup y = x \wedge y \neq x$ (for $n \geq 1$)

Note: $|x| = 0 \vee \ldots \vee |x| = n - 1 \vee |x| \geq n$ is not valid

Axioms of union

U1

$$\exists^{=1} x\ |x| = 0$$

U2 For $n, k \in \omega$,

$$\exists^{\geq n} x\ (x = \bigcup^{k} x \wedge \bigwedge_{1 \leq l < k} x \neq \bigcup^{l} x)$$

U3 For $n \geq 1$,

$$\forall x\ (\bigvee_{i=0}^{n-1} |x| = i \vee |x| \geq n)$$

Axioms of union

U4 For $n, k \geq 1$,

$$\forall x \left(|x| = n \rightarrow x \neq \bigcup^k x\right)$$

U5 For $n, m \in \omega$,

$$\forall x \left(|x| \geq n \rightarrow \exists^{\geq c(n,m)} y \left(\bigcup y = x \wedge |y| = m\right)\right)$$

U6

$$\forall x \exists^{=1} y \left(\bigcup y = x \wedge |y| = 1\right)$$

Quantifier elimination

Let $\mathcal{L}^*_U = \{\bigcup, \emptyset, |x| = n, |x| \geq n\}$. Let Λ^*_U be constituted by U1 - U6, Dn s, En s, and $|\emptyset| = 0$

Lemma

For every $\mathcal{L}^*_U$-formula θ, there is a quantifier-free $\mathcal{L}^*_U$-formula ψ such that

$$\Lambda^*_U \vdash \theta \leftrightarrow \psi$$

Corollary

$\Lambda_U =$ U1 - U6 is complete

Quantifier elimination

Example

φ is

$$\bigcup x = t \wedge x \neq s_0 \wedge \ldots \wedge x \neq s_m \wedge |x| = n,$$

where $t, s_0, \ldots, s_m$ are terms do not contain x

We call s_i a competitor if $\bigcup s_i = t = \bigcup x$, a competitor is a threat if $|s_i| = n$. Then $\exists x \varphi$ is equivalent to

$$\bigvee_{k=\lceil \log n \rceil}^{k'} \left|\left\{ s_i \;\middle|\; \bigcup s_i = t \wedge |s_i| = n \right\}\right| < c(k, n) \wedge |t| \geq k.$$

Quantifier elimination

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More reducts

Theorem

$(V, \bigcap)$ has a decidable theory:

I1 $\exists^{=1} x \bigcap x = x$

I2 $\forall x (\bigcap^k x = x \to \bigcap x = x)$ (for $k > 1$)

I3 $\forall x \exists^{\geq n} y \bigcap y = x$ (for $n < \omega$)

More reducts

Theorem

$(V, \bigcup, P)$ has a decidable theory: U1 - U6 together with

P1 $\forall x\, (|Px| \ngeq n \to \bigvee_{m \leq \lfloor \log n \rfloor} |Px| = 2^m)$ (for $n < \omega$)

P2 $\forall x\, (|x| = n \leftrightarrow |Px| = 2^n)$ (for $n < \omega$)

P3 $\forall x \bigcup Px = x$;

P4 $\forall x\, (P^l x \neq \bigcup^k x)$ (for $l > 1$ and $k < \omega$)

Dichotomy among reducts of $(V, \in)$

$\in$-complete:

$(V, \subset, \{x\})$, $(V, \subset, \bigcup)$, $(V, \subset, \bigcap)$, $(V, \bigcup, \bigcap)$, $(V, \subset, P)$ etc.

completely axiomatizable:

$(V, \subset)$, $(V, \subset, \subset^*)$, $(V, \bigcup)$, $(V, \bigcap)$, $(V, \bigcup, P)$, etc.

Dichotomy among reducts of $(V, \in)$

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Further Observation

Definition

- A stratification of a set theory formula φ is a function σ on the variables occurring in φ to the natural numbers such that
  - if $x = y$ occurs in φ, then $\sigma(x) = \sigma(y)$
  - if $x \in y$ occurs in φ, then $\sigma(x) + 1 = \sigma(y)$
- A formula φ is stratifiable if there is a stratification such that all free variables in φ have the same σ value.

Further Observation

Observation

- $X \subset Y$, $X \cup Y$, $X \cap Y$, $X \setminus Y$, $X \triangle Y$, $|X| = \infty$, $X \sim Y$ can be defined by a stratifiable formula
- while, $\bigcup X$, $\bigcap X$, $P(X)$ are not stratifiable.

Further Observation

Observation (McKenzie)

If relations and functions in reduct $(V, R_1, \ldots, R_n, f_1, \ldots, f_m)$ are all stratifiable, then it is not $\in$-complete

Question (McKenzie and Y.)

Must every stratifiable $(V, R_1, \ldots, R_n, f_1, \ldots, f_m)$ admits a complete axiomatization? For instance, $(V, \subset, \sim)$?

Thank you!