Reverse functional analysis on complex Hilbert spaces

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A countable vector space A over $\mathbb{Q} + i\mathbb{Q}$ consists of a set $|A| \subseteq \mathbb{N}$ with operations $+, \cdot$ and distinguished element $0 \in |A|$ such that $(|A|, +, \cdot, 0)$ satisfies the usual properties of a vector space over $\mathbb{Q} + i\mathbb{Q}$.

Definition 1 (RCA_0)

A (complex separable) Hilbert space H consists of a countable vector space A_H over $\mathbb{Q} + i\mathbb{Q}$ together with a function $(,) : A_H \times A_H \to \mathbb{C}$ satisfynig

(1) $(x, x) \geq 0$, $(x, y) = (y, x)$

(2) $(ax + by, z) = a(x, z) + b(y, z), (x, y) = \overline{(y, x)}$

for all $x, y, z \in A_H$ and $a, b \in \mathbb{Q} + i\mathbb{Q}$.

An element x of H is a sequence $\langle x_n : n \in \mathbb{N} \rangle$ from A_H such that $||x_n - x_m|| = \sqrt{< x_n - x_m, x_n - x_m >} \leq 2^{-n}$ whenever $n \leq m$.

Let H be a Hilbert space. A closed subspace M is defined as a separably closed subset of H, i.e, it is defined by a sequence $\langle x_n : n \in \mathbb{N} \rangle$ from H such that $x \in M$ if and only if for any $\varepsilon > 0$, $||x - x_n|| < \varepsilon$ for some n.

Theorem 2 (RCA_0, Avigad and Simic 06)

Each of the following statements is equivalent to ACA:

(1) *For every closed subspace M of a Hilbert space H, the orthogonal projection P_M for M exists.*

(2) *For every closed subspace M of H and every point x in H, the orthogonal projection of x on M exists.*

(3) *For every closed subspace M of H and every point x in H, $d(x, M)$ exists.*

For a subset A of H, $x \in A^{\perp}$ is an element such that $(x, y) = 0$ for all $y \in A$.

Theorem 3 (RCA_0, Tanaka and Saito 96?)

The following statement is equivalent to ACA: For every closed subspace M of a Hilbert space H, a closed subspace $M^{\perp}$ exists.

Note that if $M^{\perp}$ may not exist, we can state $H = M \oplus M^{\perp}$ by $\mathcal{L}_2$-formula. From Theorem 2, this holds.

Proposition 4 (RCA_0)

The following statement is equivalent to ACA: For every closed subspace M of a Hilbert space H, $H = M \oplus M^{\perp}$

Theorem 5 (RCA_0, Avigad and Simic 06)

Any Hilbert space has an orthonormal basis.

So two infinite dimensional Hilbert spaces are unitarily equivalent. Let $\langle e_n : n \in \mathbb{N} \rangle$ be an orthonormal basis of H. We have Parseval's identity:

$$||x||^2 = \sum_{n=0}^{\infty} |a_n|^2 \qquad \text{where } a_n = (x, e_n).$$

Definition 6 (RCA_0)

A bounded linear operator T between Hilbert spaces H_1 and H_2, is a function $T : A_{H_1} \to H_2$ such that

(1) *T is linear, i.e., $T(q_1x_1 + q_2x_2) = q_1T(x_1) + q_2T(x_2)$ for all $q_1, q_2 \in \mathbb{Q} + i\mathbb{Q}$ and $x_1, x_2 \in A_{H_1}$.*

(2) *The norm of T is bounded, i.e., there exists a real number K such that $||T(x)|| \leq K||x||$ for all $x \in A_{H_1}$.*

Then, for $x = \langle x_n : n \in \mathbb{N}\rangle \in H_1$, we define $T(x) = \lim_{n\to\infty} T(x_n)$. So we can regarded T as $T : H_1 \to H_2$.

A linear operator $T : H_1 \to H_2$ is bounded if and only if it is continuous. A linear functional T is a linear operator from a Hilbert space H to $\mathbb{C}$.

The Riesz representation theorem is the statement that any bounded linear functional T on a Hilbert space H, has a unique vector $y \in H$ such that $T(x) = (x, y)$ for each $x \in H$.

Fact 7 (RCA_0,Tanaka and Saito 96?)

The Riesz representation theorem is equivalent to ACA.

The proof is simple. To prove the Riesz representation theorem implies ACA, for an injective function $f : \mathbb{N} \to \mathbb{N}$, consider $T : l^2 \to \mathbb{C}; e_n \mapsto \sum_{i<n} 2^{-f(i)}$. Take $y \in l^2$ such that $T(x) = (x, y)$ for each $x \in l^2$, then $||y|| = \sum_{n=0}^{\infty} 2^{-f(n)}$. $\square$

Let $\langle x_n : n \in \mathbb{N} \rangle$ be a sequence from H and $x \in H$. Define

1. $x_n \to x$ (w) $\Leftrightarrow (x_n, y) \to (x, y)$ for all $y \in H$.
2. $x_n \to x$ (s) $\Leftrightarrow \lim_{n\to\infty} ||x_n - x|| = 0$.

Proposition 8 (RCA_0)

(1) *$x_n \to x$ (w) and $||x_n|| \to ||x||$, then $x_n \to x$ (s)*
(2) *$x_n \to x$ (w) and $y_n \to y$ (s), then $(x_n, y_n) \to (x, y)$*

To prove (2), we use the Uniform boundedness principle which is proved in RCA_0.

Proposition 9 (RCA_0)

The following statement is equivalent to ACA: any bounded sequence $\langle x_n : n \in \mathbb{N} \rangle$ from a Hilbert space has a weakly convergent subsequence.

For a bounded linear operator $T : H_1 \to H_2$, $T^* : H_2 \to H_1$ is the adjoint if $(Tx, y) = (x, T^*y)$ for all $x \in H_1$ and $y \in H_2$.

Theorem 10 (RCA_0, Tanaka and Saito 96)

The existence of the adjoint for any bounded linear operator is equivalent to ACA.

In fact, the following statement already implies ACA: For any bounded linear operator $T : l^2 \to l^2$ and any $x \in l^2$, there exists $u \in l^2$ such that $(Ty, x) = (y, u)$ for all $y \in l^2$.

Basic properties of the adjoint, if it exists, are shown in RCA_0.

Let $\langle T_n : n \in \mathbb{N} \rangle$ be a sequence of bounded linear operators from H_1 to H_2, and T a bounded linear operator from H_1 to H_2. Define

1. $T_n \to T\ (w) \Leftrightarrow T_n(x) \to T(x)\ (w)$ for all $x \in H_1$.
2. $T_n \to T\ (s) \Leftrightarrow T_n(x) \to T(x)\ (s)$ for all $x \in H_1$.
3. $T_n \to T$ uniformly $\Leftrightarrow$ there is a sequence $\langle r_n : n \in \mathbb{N} \rangle$ of nonnegative reals such that $||T_n(x) - T(x)|| \leq r_n$ for all n and $x \in H_1$ and $\lim_n r_n = 0$.

Let $T_n, T : H_1 \to H_2$ and $S_n, S : H_2 \to H_3$.
If $T_n \to T\ (s)$ and $S_n \to S\ (s)$, then $S_n T_n \to ST\ (s)$.
If $T_n \to T\ (w)$ and their adjoints exist, then $T_n^* \to T^*\ (w)$.
These and the uniform-continuity versions are proved in RCA_0.

Theorem 11 (Banach-Steinhaus Theorem)

Let H_1 and H_2 be Hilbert spaces. Let $\langle T_n : n \in \mathbb{N} \rangle$ be a sequence of bounded linear operators from H_1 to H_2. If $\langle (T_n x, y) : n \in \mathbb{N} \rangle$ is convergent for any $x, y \in H_1$, then there exists a bounded linear operator $T : H_1 \to H_2$ such that $T_n \to T$ (w).

Theorem 12 (RCA_0)

The Banach-Steinhaus theorem is equivalent to ACA.

For self-adjoint operators T_1 and T_2 over H, $T_1 \leq T_2$ if $(T_1 x, x) \leq (T_2 x, x)$ for all $x \in H$. If $O \leq T$, then $O \leq T^n$, and if $O \leq T \leq I$, then $T^n \leq T^m$ for $m \leq n$, by the usual induction.

Using the above version of the Banach-Steinhaus theorem, we can show this.

Theorem 13 (RCA_0)

The following statement is equivalent to ACA: Let $\langle T_n : n \in \mathbb{N} \rangle$ be an increasing sequence of self-adjoint operators bounded some self-adjoint operator S. Then it strongly converges to some self-adjoint operator T.

For a closed subspace M, if the orthogonal projection P_M exists, P_M is a positive self-adjoint operator which is idempotent. Conversely, given an idempotent self-adjoint operator P, we define a closed subspace M by $\langle P(a) : a \in A_H \rangle$. Then $P = P_M$.

Theorem 14 (RCA_0)

Each of the following statements is equivalent to ACA:

(1) *Any increasing sequence $\langle P_n : n \in \mathbb{N} \rangle$ strongly converges to some projection.*

(2) *Any decreasing sequence $\langle P_n : n \in \mathbb{N} \rangle$ strongly converges to some projection.*

A bounded linear operator $U : H \to H$ is an isometry if $||U(x)|| = ||x||$ for all $x \in H$. A surjective isometry is said to be unitary.

A bounded linear operator $U : H' \to H$ is a "partial" isometry on H if $||U(x)|| = ||x||$ for all $x \in H'$.

Proposition 15 (RCA_0)

The following statement is equivalent to ACA: For any bounded linear operator T of a Hilbert space H, there are a positive self-adjoint Q and a "partial" isometry U such that $||Qx|| = ||Tx||$ for all $x \in H$ and $T = UQ$.

if T is normal, that is, $T^*T = TT^*$, then the above U can be taken as unitary, as usual.

The idea of the proof. For an injective function $f : \mathbb{N} \to \mathbb{N}$, consider $T : l^2 \to l^2; e_0 \mapsto e_0, e_n \mapsto 2^{-f(n-1)/2} e_0 \ n > 0$. Then $||Q^2 e_0||^2 = 1 + \sum_{n=0}^{\infty} 2^{-f(n)}$. □

We say that a bounded operator T on H is invertible if T is a bijection of H and its inverse is also bounded. The spectrum of T, denoted by $\sigma(T)$, is the set of complex numbers z for which $T - zI$ is not invertible.

Proposition 16 (RCA_0)

If T is self-adjoint, then $\sigma(T)$ is a bounded subset of reals.

ACA_0 implies $\sigma(T)$ is closed.

Proposition 17 (Π^1_1-CA_0)

Any compact self-adjoint operator T has a sequence $\langle P_n : n \in \mathbb{N} \rangle$ of projections and a sequence $\langle r_n : n \in \mathbb{N} \rangle$ of real numbers such that $P_n P_m = 0$ for any $n \neq m$ and $\lim_{n\to\infty} r_n = 0$ and $T_n = \sum_{i<n} r_i P_i \to T$ uniformly.

References

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- Jeremy Avigad and Ksenija Simic, *Fundamental Notions of Analysis in Subsystems of Second-Order Arithmetic*, Annals of Pure and Applied Logic, 139:138-184, 2006.