The Eigen distribution for multi-branching game trees on ID

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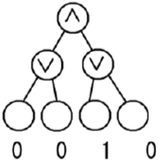
CTFM, March 21-27, 2019, WUT

AND-OR trees

An AND-OR tree is a tree whose root is labeled by AND and nodes are level-by-level labeled by OR or AND alternatively except for leaves.

Each leaf is assigned Boolean value 1 or 0, where 1 denotes true and 0 denotes false.



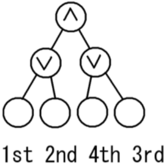
$$f(0,0,1,0) = (0 \vee 0) \wedge (1 \vee 0) = 0$$

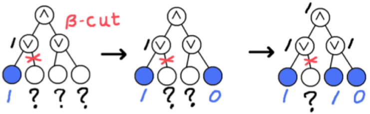
α-β pruning algorithms

An α-β pruning algorithm satisfies the following conditions.

- When an algorithm knows a child of an AND-node has value 0, it recognizes that the value of AND-node is 0 without probing other children (α-cut).
- When an algorithm know a child of an OR-node has value 1, it recognizes that the value of OR-node is 1 without probing other children (β-cut).

α-β pruning algorithm $A = 1243$.





$C(A, \omega)$: the number of leaves checked by A under assignment ω.

Distributions on assignments

Let d be a (probability) distribution on the set Ω of assignments, the expected cost of algorithm A under the distribution d is defined by

$$C(A,d) := \sum_{\omega \in \Omega} C(A,\omega)d(\omega).$$

Let D be the set of distributions and $\mathcal{A}$ the set of deterministic algorithms computing tree T.

The *distributional complexity* computing tree T is defined by

$$\max_{d \in D} \min_{A \in \mathcal{A}} C(A,d).$$

A distribution d is said to be an *eigen-distribution* if

$$\min_{A \in \mathcal{A}} C(A,d) = \max_{d' \in D} \min_{A \in \mathcal{A}} C(A,d').$$

Background

- Saks and Wigderson (1986) showed that the randomized complexity of n-branching trees with height h is $\Theta\left(\left(\frac{n-1+\sqrt{n^2+14n+1}}{4}\right)^h\right)$.
- Yao's Principle (1977) implies that the **randomized complexity** equals to the **distributional complexity**.

$$\underbrace{\min_{A_R \in \mathcal{A}_R} \max_{\omega} C(A_R, \omega)}_{\text{Randomized complexity}} = \underbrace{\max_{d} \min_{A \in \mathcal{A}} C(A, d)}_{\text{Distributional complexity}}.$$

where $\mathcal{A}_R$ denotes the class of probability distribution over the family of deterministic algorithms.

IID

- D:= the set of all distributions
- ID:= the set of all independent distributions.
- IID:= the set of all independent identical distributions.

Let Ω be the set of assignments for a given tree. We say $d : \Omega \to [0,1]$ is an *independent distribution* (denoted by $d \in$ ID) if there exist p_i's (the probability of the i-th leaf that has value 0) such that for any $\omega \in \Omega$,

$$d(\omega) = \prod_{\{i:\ \omega(i)=0\}} p_i \prod_{\{i:\ \omega(i)=1\}} (1 - p_i).$$

We say $d \in$ IID if d is an ID satisfying $p_1 = p_2 = \cdots = p_n$.

T_2^h

	eigen-distribution is unique (D)	$d \in ID \to d \in IID$
T_2^h	• Liu-Tanaka (2007) • False w.r.t. only directional-alg by Suzuki and Nakamura (2013)	• Claimed by Liu-Tanaka (2007) • Justified by Suzuki and Niida (2015)

Question

How about multi-branching trees, especially T_n^h, Balanced Multibranching trees?

A tree is *balanced* if each nonterminal node at the same level has the same number of children.

Note that we do not require that nodes from different levels have the same number of children

The n-branching tree with height h is denoted by T_n^h.

T_n^h(Balanced Multi-branching trees)

	eigen-distribution is unique (D)	$d \in ID \to d \in IID$
T_2^h	• Liu-Tanaka (2007) • False w.r.t. only directional-alg by Suzuki and Nakamura (2013)	• Claimed by Liu-Tanaka (2007) • Justified by Suzuki and Niida (2015)
T_n^h	• Holds w.r.t. deterministic-alg • by Peng et al. (2016)	• Holds if we restrict $0 < r < 1$ by Peng et al (2017)

Remark: r is the probability of root being 0.

Let ID(r) denote the set of independent distributions which induce that the probability of the root having value 0 is r.

Theorem (Peng *et al.* (2017))

For any balanced multi-branching AND-OR tree $\mathcal{T}$, we fix $\delta \in \mathrm{ID}(r)$ and $0 < r < 1$. If the following equation holds,

$$\min_{A:\mathrm{depth}} C(A, \delta) = \max_{d \in \mathrm{ID}(r)} \min_{A:\mathrm{depth}} C(A, d),$$

then $\delta \in \mathrm{IID}$.

We can show the following conclusion which is a generalization of Suzuki-NIida's result in Suzuki (2015).

Theorem (1)

Suppose that T is an n-branching AND-OR tree (OR-AND tree). Let $r \in \{0,1\}$, d_0 be the IID such that each leaf has the probability $1-r$. Then,

- *in the case where the height of T is even, denoted by $h = 2k$, $\min_A C(A, d_0) = n^k$.*
- *In the case where h is odd, denoted by $2k+1$,*
  - $\min_A C(A, d_0) = n^k$ *if we consider AND-OR tree and $r = 0$, or we consider OR-AND tree and $r = 1$.*
  - $\min_A C(A, d_0) = n^{k+1}$, *otherwise.*

ID implies IID

Theorem (Peng, unpublished)

For any n-branching tree T, suppose that d_1 is an ID such that the following holds.

$$\min_A C(A, \delta) = \max_d \min_A C(A, d),$$

where d over all IDs and A over all depth-first algorithms. Then δ is an IID.

Sketch of proof: This theorem holds in the case $0 < r < 1$ by Peng *et al.* (2017), the left work is to investigate the case $r = 0$ and $r = 1$. It is enough to show the following claim.

Claim: When $r = 0$ or 1, there exists r_0 such that

$$\min_A C(A, \delta) < \max_{d \in ID_{r_0}} \min_A C(A, d),$$

where $\delta \in IID$ such that probability of the root is r, and d over all IDs such that the probability of the root is r_0.

Proof of Claim: Let x be the probability of each leaf having value 0, r_x be the probability of the root having value 0 with respect to x. Given an $d \in IID$, for any depth-first algorithm A, we get the same expected cost. i.e., $\min_{A_0} C(A_0, d) = C(A, d)$.

Let $x = 1/2$, it is clear that for any depth-first algorithm A,

$$C(A, d_{1/2}) \leq \max_{d \in ID_{r_{1/2}}} \min_A C(A, d). \qquad (*)$$

We also can show the following conclusions:

- In the case $h = 2k$, $C(A, d_{r_{1/2}}) > n^k$,
- In the case where $h = 2k + 1$,
  - $C(A, d_{r_{1/2}}) > n^k$ if we consider AND-OR tree and $i = 0$, or we consider OR-AND tree and $i = 1$.
  - $C(A, d_{r_{1/2}}) > n^{k+1}$, otherwise.

By Theorem 1, $\min_A C(A, \delta) < C(A, d_{r_{1/2}})$ holds. we complete the proof of the claim. □

Non-depth-first-algorithms

With the condition $0 < r < 1$, Suzuki(2018) extended our results to the case where non-depth-first algorithms are taken into consideration.

Definition (Depth-first-algorithms)

if an algorithm evaluates the value of one subtree, it will never evaluate the others until it completes the current one.

Otherwise, it is called **non-depth-first-algorithm**.
Thus, the above theorem still holds with respect to non-depth-first algorithms.

Weighted trees on ID case

Definition (Okisaka *et al.* 2017)

Let A be an algorithm, ω an assignment, $\sharp_1(A, \omega)$ (resp., $\sharp_0(A, \omega)$) denote the number of leaves probed by A and assigned 1 (resp., 0) on ω. For any positive real numbers a, b,

$$C(A, \omega; a, b) := a \cdot \sharp_1(A, \omega) + b \cdot \sharp_0(A, \omega),$$

is called a *generalized cost* weighted with (a, b). Obviously, $C(A, \omega) = C(A, \omega; 1, 1)$.
For a distribution d on Ω, the *expected generalized cost* $C(A, d; a, b) := \sum_{\omega \in \Omega} d(\omega) \cdot C(A, \omega; a, b)$. We may say that $\mathcal{T}$ is a tree weighted by (a, b) if we consider the generalized cost.

Note that: for weighted trees, the weight not dependent on the assigned value.

We can show the following conclusions:

Theorem (Peng *et al.* unpublished)

For any multi-branching weighted tree, a directional algorithm w.r.t. ID *is optimal.*

We consider an IID on $\mathcal{T}_n^1(a, b)$ such that each leaf is assigned 0 with probability x. The expected cost is denoted by $C(x, a, b)$.

Lemma (Peng *et al.* unpublished)

Suppose that the distribution on $\mathcal{T}_n^1$ weighted with (a, b) is an IID *with all leaves assigned probability x. Then*

(1) $p(x)$ *is a strictly increasing function of x.*

(2) $\dfrac{C(x, a, b)}{p(x)}$ *is strictly decreasing.*

(3) $\dfrac{C'(x, a, b)}{p'(x)}$ *is strictly decreasing.*

Theorem (Peng *et al.* unpublished)

For any balanced multi-branching AND-OR tree $\mathcal{T}$ weighted by (a, b), we fix $\delta \in \mathrm{ID}(r)$ and $0 < r < 1$. If the following equation holds,

$$\min_{A:\mathrm{depth}} C(A, \delta, a, b) = \max_{d \in \mathrm{ID}(r)} \min_{A:\mathrm{depth}} C(A, d, a, b),$$

then $\delta \in \mathrm{IID}$.

Thank you for your attention!