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Models of Computations
—Information Systems and Domains

Luoshan Xu

School of Mathematics Science

Yangzhou University

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1 Introduction

Computations

- can be viewed as both functions and process.
- can be carried out by programs.

- are changes of states (of Turing machines).
- can be taken as maps from Input information to Output information.
- can also be taken as modal logic of inferences (formula), special binary relations, partial orders.

- So, to study computations is to study posets, and study states of information systems, what we should study for posets?
- In order to assign meanings to programs written in high-level programming languages, Dana Scott invented continuous lattices [14] which is now grown up as Domain Theory [1, 4].

- From states of computations, with continuity, domains can be taken as models of denotational semantics of computations.

How to model computations? —By domains:

- Structures arising in theoretical computer science admit natural partial orders of appropriate information content.
- The more information some state contains, the larger it is in the information order.
- It is a common sense that the increasing sequence of information should give more (converges to) accurate states (of computation).
- D. Scott lead to the discovery (1972): continuous lattices [14], now more generalized as domains = continuous dcpos.

- Domain theory is one of the important research fields of theoretical computer science. Mutual transformations and infiltration of order, topology and logic are the basic features of this theory.

- Ways to characterize domains: not only by continuity, but also by
  Stone duality [3],
  abstract bases [24],
  formal topologies [25],
  information systems [2, 16],
  rough approximable concepts [6] and
  F-augmented closure spaces [5].

How to model computations? —By information systems:

- From actions of a computation, Dana Scott in his seminal paper [15], introduced information systems as a logic-oriented approach to denotational semantics of programming languages, or, models of denotational semantics of computations.

- A large volume of work followed with information systems has been done [8, 9, 18, 19, 20, 21, 26, 27, 28].

- In 1993, Hoofman [8] introduced continuous information systems (shortly, cis) in his sense that represent bc-domains (the continuous counterpart of Scott domains).

- In 2001, Bedregal [2] modified Hoofman's definition of cis.

- In 2008, Spreen, Xu and Mao [16] first introduced a new concept of continuous information systems (in short, C-inf). C-infs generate/represent exactly all the continuous (not necessarily pointed) domains.

- Later, Xu and Mao [26] introduced the concept of algebraic information system (in short, A-inf).
- In 2012, Spreen in [17] introduced L-information systems which represent all pointed L-domains.
- In 2013, Wu and Li [20] proposed new algebraic information systems (e-quivalent to A-infs) with briefer conditions to represent algebraic domains.

- In 2016, by adding new conditions to C-infs, Wu, Guo and Li [21] provid-ed a kind of information systems which serve as representations of general L-domains.

- Since domains and information systems are all models of computations, they are closely linked.
- We will see that

  —- all the states of a C-inf forms a domain, and

  —- every domain can induce an information system in a standard manner.

We for details, in this talk,

- introduce basic concepts for domains and information systems.
- introduce results for domains and information systems.
- give relationships of the two kinds of models.
- and propose some further topics.

Some of them are newly obtained by our group.

2 Continuities of posets and Scott topology

One of the important things for posets is the way-below relation, or approximation order.

Definition 2.1. (Way-below relation) Let P be a poset, $x,\ y \in P$. We say that x approximates y, written $x \ll y$, if whenever D is directed with $\sup D \geqslant y$, then $x \leqslant d$ for some $d \in D$. We use $\Downarrow x$ to denote the set $\{a \in P : a \ll x\}$.

If for every element $x \in P$, the set $\Downarrow x := \{a \in P : a \ll x\}$ is directed and $\sup \Downarrow x = x$, then P is called a *continuous poset*. A continuous poset which is also a dcpo (resp., bounded complete dcpo, complete lattice) is called a *continuous domain* or briefly a domain (resp., bc-domain, continuous lattice).

“$\ll$” also known as way-below relation.

Example 2.2. Some examples and counterexamples:

- Continuous posets: discrete sets, (0, 1), $\mathbb{R}$, $\mathbb{N}$, ...
- Domains: half open unit interval (0,1], finite posets, ...
- Continuous lattices: CD-lattices, topologies of compact Hausdroff spaces.
- NOT continuous: complete lattice shaped “$\diamondsuit$” .

Definition 2.3. Let P be a poset, $B \subseteq P$. The set B is called a basis for P if $\forall a \in P$, there is a directed set $D_a \subseteq B$ such that $\forall d \in D_a$, $d \ll_P a$ and $\sup_P D_a = a$.

Theorem 2.4. *A poset P is continuous iff it has a basis.*

To clarify relationships of continuous posets and domains, the concept of embedded basis for posets is useful.

Definition 2.5. (Xu, 2006, [23]) Let B and P be posets. If there is a map $j : B \to P$ satisfying
(1) j preserves existing directed sups,
(2) $j : B \to j(B)$ is an order isomorphism,
(3) $j(B)$ is a basis for P,
then (B, j) is called an embedded basis for P. If $B \subseteq P$ and (B, i) is an embedded basis for P, where i is the inclusion map, then we say also that B is an embedded basis for P.

Theorem 2.6. *(Xu, 2006, [23]) A poset P is continuous iff there is a domain $\hat{P}$ such that P is an embedded basis of $\hat{P}$. Here, $\hat{P}$ is a directed completion of P.*

Definition 2.7. Let P be a poset and $A \subseteq P$. If $\downarrow A = A$ and, for any directed set $D \subseteq A$, $\sup D \in A$ if $\sup D$ exists, then A is called Scott-closed. The complements of the Scott-closed sets form a topology, called the Scott topology, denoted $\sigma(P)$.

A remarkable characterization of continuous posets by topology is

Theorem 2.8. [13, 23] *A poset is continuous if and only if the lattice of its Scott closed sets is a completely distributive complete lattice (CD-lattice).*

A very useful property of a continuous poset is

Proposition 2.9. (see [4]) *If P is a continuous poset, then the interpolation property holds:* (INT)*:* $x \ll z \Rightarrow \exists y \in P$ *such that* $x \ll y \ll z$.

Definition 2.10. A map $f : P \to Q$ is called *Scott continuous* if it is continuous with respect to the Scott topologies.

$[P \to Q]$: the function space=the poset of all Scott continuous maps with the pointwise order.

Lemma 2.11. *Let P, Q be a posets. Then a map $f : P \to Q$ is Scott continuous iff f preserves existing directed sups.*

The category of domains and Scott continuous functions **DOM** is not Cartesian closed (ccc). So,

Achim Jung, 1988, introduced some special kinds of domains.

- FS-domains,
- L-domains,
- B-domains
- algebraic FS-domains = Bifinite domains.

It is known that (Achim Jung)

- The category of FS-domains and Scott continuous functions **FSDOM** is a Cartesian closed category (ccc), and is maximal in **DOM**.
- The category of L-domains and Scott continuous functions **L-DOM** is a Cartesian closed category (ccc), and is maximal in **DOM**.
- The category of B-domains and Scott continuous functions **BDOM** is a Cartesian closed category (ccc), and is a subcategory of **FSDOM**.

Now many kinds of continuities of posets have been introduced from mathematical points of view and their relations are discussed.

- Xu&Mao: hyper continuity (stronger), by upper topology,
- Zhou&Zhao: supcontinuity (similar to), by arbitrary unions,
- Ho&Zhao: C-continuity (similar to), by Scott closed sets,
- Lawson, Xu, etc.: quasicontinuity (weaker), by approximation order of subsets,
- Bai: uniform continuity (similar to), by uniform sets,
- Kou, Xu&Mao: meet continuity (weaker), by Scott open sets,
- Li&Zhang: θ-continuity (similar to), by some kind of cuts.

Some useful characterizations/relations of continuities are

- A poset is quasicontinuous iff its Scott topology is a hyper continuous lattice;
- A poset is meet continuous iff its Scott closed sets form a cHa;
- A poset is continuous iff it is meet continuous and quasicontinuous;
- A poset is hyper continuous iff it is continuous and its Scott topology is the upper topology;
- A poset is supercontinuous iff every two different points can be separated by a principal filter and the complement of a Scott S-set, iff every two different points can be separated by a Scott S-set filter.

3 Continuous information systems

Recall that Huang, He&Xu in [7, 9], an *information structure* is a triple $(A, Con, \vdash)$, where

- A is a set and the elements of A are usually called *tokens*,
- Con is a family of some finite subsets of A, and are *consistent* in meaning.
- $\vdash \subseteq Con \times A$ is a relation called an *entailment relation*.
- use $B \subseteq_{\text{fin}} A$ to denote that B is a finite subset of A,
- use $X \vdash b$ to mean that $(X, b) \in \vdash$, read that from consistent X, one can deduce/compute b,
- use $X \vdash F$ to mean $F \subseteq_{\text{fin}} \{b \in A : X \vdash b\}$,
  or equivalently, F is finite and $X \vdash b$ for all $b \in F$.

Definition 3.1. [15] An information structure $\mathcal{S} = (A, Con, \vdash)$ is called a *Scott information system* (in short, Scott-inf) if the following six conditions hold for any sets $X, Y \in Con$, $a \in A$:

(S1) $\emptyset \in Con$,

(S2) $(Y \subseteq X \in Con) \Rightarrow (Y \in Con)$,

(S3) $\{a\} \in Con$,

(S4) $(X \vdash a) \Rightarrow X \cup \{a\} \in Con$,

(S5) $(\forall a \in X \in Con)(X \vdash a)$,

(S6) $X \vdash Y \wedge Y \vdash a \Rightarrow X \vdash a$.

Definition 3.2. [16, 26, Spreen, Xu, Mao] An information structure $\mathcal{S} = (A, Con, \vdash)$ is called a *continuous information system* (in short, C-inf) if the following six conditions hold for any sets $X, Y \in Con$, $a \in A$ and **nonempty** finite subset $F \subseteq A$:

(1) $\{a\} \in Con$,

(2) $X \vdash a \Rightarrow X \cup \{a\} \in Con$,

(3) $(Y \supseteq X \wedge X \vdash a) \Rightarrow Y \vdash a$,

(4) $X \vdash Y \vdash a \Rightarrow X \vdash a$,

(5) $X \vdash a \Rightarrow (\exists Z \in Con)(X \vdash Z \wedge Z \vdash a)$,

(6) $X \vdash F \Rightarrow (\exists Z \in Con)(Z \supseteq F \wedge X \vdash Z)$.

If in addition, $\mathcal{S}$ satisfies

(S5) $\forall a \in X \in Con, X \vdash a$

in Definition 3.1, then $\mathcal{S}$ is called an *algebraic information system* (in short, A-inf).

It is easy to see that Scott-infs are A-infs.

Definition 3.3. [9, 16] Let $\mathcal{S} = (A, Con, \vdash)$ be an information structure. A subset $x \subseteq A$ is a *state* of $\mathcal{S}$ if the following three conditions hold:

(1) (finitely consistency) $(\forall F \subseteq_{\text{fin}} x)(\exists Y \in Con)(F \subseteq Y \wedge Y \subseteq x)$,

(2) ($\vdash$ closedness) $(\forall X \in Con)(\forall a \in A)(X \subseteq x \wedge X \vdash a \Rightarrow a \in x)$,

(3) (derivability) $(\forall a \in x)(\exists X \in Con)(X \subseteq x \wedge X \vdash a)$.

With respect to the order of set inclusion $\subseteq$, the states of an information structure $\mathcal{S}$ form a partially ordered set, denoted by $|\mathcal{S}|$.

Proposition 3.4. *He&Xu [7] Let $\mathcal{S} = (A, Con, \vdash)$ be an information structure and $\{x_i : i \in I\}$ a directed set of $|\mathcal{S}|$. Then $\vee_{|\mathcal{S}|}\{x_i : i \in I\} = \cup_{i \in I} x_i$, and thus $|\mathcal{S}|$ is a dcpo.*

Theorem 3.5. [16, Theorem 20] *Let* $\mathcal{S} = (A, Con, \vdash)$ *be a C-inf. Then* $|\mathcal{S}|$ *is a domain.*

From a domain D, a C-inf can be induced with the method given in [26].

Definition 3.6. Xu&Mao[26] For a domain D with a basis B, define an information structure $\mathcal{S}(D, B)$=$(B, Con_D, \vdash_D)$ such that

(1) $X \in Con_D \Leftrightarrow X \subseteq_{\text{fin}} B$ and $\vee X$ exists in D;

(2) $\forall X \in Con_D, \forall b \in B, X \vdash_D b \Leftrightarrow b \ll \vee X.$

Theorem 3.7. *Let D be a domain with a basis B. Then $\mathcal{S}(D, B)$ defined above is a C-inf, called the* induced C-inf *by domain D with basis B.*

It should be noted that there are different manners to induce continuous information systems.

To get information structures from a given domain, one can obtain many different C-infs. Some of them may has particular property which we will see later.

Lemma 3.8. [26, Theorem 2.4] (1) *For a domain D with a basis B, $\mathcal{S}(D, B) = (B, Con_D, \vdash_D)$ is indeed a C-inf. Furthermore, $|\mathcal{S}(D, B)| \cong D$. In particular, $|\mathcal{S}(D, D)| \cong D$.*
(2) *Let D be an algebraic domain with $K(D)$ the set of all compact elements. Then the induced C-inf $\mathcal{S}(D, K(D)) = (K(D), Con_D, \vdash_D)$ in the sense of Definition 3.6 is an A-inf.*

Definition 3.9. Let D be a poset and $\mathcal{S}$ an information structure. If $|\mathcal{S}| \cong D$, then $\mathcal{S}$ is called a *representation* of D, or $\mathcal{S}$ represents D, or D is represented by $\mathcal{S}$.

Clearly, every information structure $\mathcal{S}$ represents $|\mathcal{S}|$.

Theorem 3.10. *A dcpo D is a domain iff D can be represented by a C-inf.*

4 Generalized algebraic information systems

Theorem 4.1. *A dcpo D is an algebraic domain iff D can be represented by an A-inf.*

To represent an algebraic domain, A-infs may not be needed.

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Proposition 4.2. *Let D be an algebraic domain. If there exists a $\xi \in D$ with $\xi \notin K(D)$, then $\mathcal{S}(D, D)$ is not an A-inf. In this case, algebraic domain D is represented by a non A-inf $\mathcal{S}(D, D)$.*

Lemma 4.3. [16, Proposition 32] *An information structure $\mathcal{S} = (A, Con, \vdash)$ is a C-inf with $|\mathcal{S}|$ being an algebraic domain iff $(A, Con, \vdash)$ satisfies Definition 3.2(1-4, 6) and the following condition*

(ALG) $(\forall X, Y \in Con)(X \vdash Y) \Rightarrow (\exists Z \in Con)(X \vdash Z \wedge Z \vdash Z \wedge Z \vdash Y)$.

So, [7, He and Xu] introduced generalized algebraic information system (in short, GA-inf).

Definition 4.4. An information structure $\mathcal{S} = (A, Con, \vdash)$ satisfies Definition 3.2(1-4, 6) and condition (ALG) in Lemma 4.3 is called a *generalized algebraic information system* (in short, GA-inf).

We immediately have the following

Proposition 4.5. (i) *Every GA-inf is a C-inf.*

(ii) *Every C-inf* $\mathcal{S} = (A, Con, \vdash)$ *with* A *being finite is a GA-inf.*

(iii) *Every A-inf is a GA-inf.*

Next counterexample shows that a GA-inf need not be an A-inf.

Example 4.6. Let $D = \mathbb{N} \cup \{\infty\}$ be a poset obtained from $\mathbb{N}$ by adjoining the largest element ∞. Clearly, D is an algebraic domain. Since ∞ is not a compact element in D, by Proposition 4.2, $\mathcal{S}(D, D)$ is not an A-inf. By Lemma 3.8(1), $\mathcal{S}(D, D)$ is a C-inf and $|\mathcal{S}(D, D)| \cong D$ is an algebraic domain, thus $\mathcal{S}(D, D)$ is a GA-inf.

Theorem 4.7. *A dcpo D is an algebraic domain iff D can be represented by a GA-inf.*

For a computation, to get the same results (state domains), one may take different actions (process, information systems). Hence this leaves ones some space to choose better behaviour (program) to carry out a computation. That reflects the significance of the study of TCS. This serves us motivation to consider a special kind of C-inf: weak algebraic information systems.

5 Weak algebraic information systems

We introduce a weak algebraic information system (in short, wA-inf), and discuss relationships among A-infs, GA-infs and wA-infs.

Definition 5.1. (cf. [2, 26]) Let $\mathcal{S} = (A, Con, \vdash)$ be an information structure. Define $w\mathcal{S}$ =$(A, Con, \models)$ such that $\forall a \in A, \forall X \in Con$,

$$X \models a \Leftrightarrow X \cup \{a\} \in Con \text{ and } (\forall b \in A, \{a\} \vdash b \Rightarrow X \vdash b).$$

Then $w\mathcal{S} = (A, Con, \models)$ is called the *induced information structure* by $\mathcal{S}$.

Next example shows the induced information structure by a C-inf need not be a C-inf.

Example 5.2. [26, Example 4.3] Let $\mathcal{S} = (A, Con, \vdash)$ be an information structure, where $A = \{1, 2, 3\}$, $Con = \mathcal{P}(A) \setminus \{2, 3\}$ and $\vdash = \{(X, 1) : 1 \in X\}$. Then it is direct to check that $\mathcal{S} = (A, Con, \vdash)$ is a C-inf. To see that $w\mathcal{S}$ is not a C-inf, we first note that $\emptyset \models 2$, $\emptyset \models 3$ and $\emptyset \subseteq \{2\}$. By Condition 3.2(3), one should have $\{2\} \models 3$, while $\{2\} \not\models 3$ for $\{2, 3\} \notin Con$. So, $w\mathcal{S}$ is not a C-inf.

Lemma 5.3. [26, Coroally 4.5] *If* $w\mathcal{S} = (A, Con, \models)$ *induced by a C-inf* $\mathcal{S}$ *is a C-inf, then* $w\mathcal{S}$ *is an A-inf.*

Definition 5.4. Let $\mathcal{S}$ =$(A, Con, \vdash)$ be a C-inf and $w\mathcal{S} = (A, Con, \models)$ the induced information structure by $\mathcal{S}$. If $w\mathcal{S}$ is a C-inf, and thus an A-inf, then $\mathcal{S}$ =$(A, Con, \vdash)$ is called a *weak algebraic information system* (in short, wA-inf).

Proposition 5.5. *Every A-inf is a wA-inf.*

Theorem 5.6. *He&Xu Every C-inf* $\mathcal{S}(D, B)$ *induced in Definition 3.6 by a domain* D *with a basis* B *is a wA-inf.*

A wA-inf need not be a GA-inf.

The following example shows that a GA-inf need not be a wA-inf, either.

Example 5.7. Let $\mathcal{S} = (A, Con, \vdash)$ be the C-inf in Example 5.2. Since $|\mathcal{S}|$ is an algebraic domain, $\mathcal{S}$ is a GA-inf. Note that $w\mathcal{S}$ induced by $\mathcal{S}$ is not a C-inf. So, $\mathcal{S}$ is not a wA-inf.

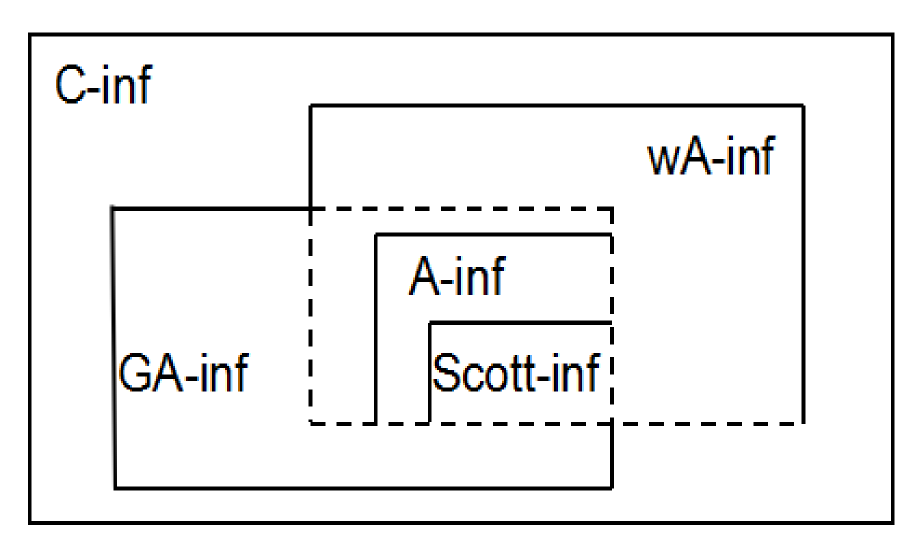
Next example shows that an information structure which is both a GA-inf and a wA-inf need not be an A-inf.

Example 5.8. Let D be the algebraic domain $\mathbb{N} \cup \{\infty\}$ in Example 4.6. Then $\mathcal{S}(D, D)$ is a GA-inf. By Theorem 5.6, $\mathcal{S}(D, D)$ is a wA-inf, while $\mathcal{S}(D, D)$ is not an A-inf by Example 4.6.

We use $A(\mathcal{S})$(resp., $wA(\mathcal{S})$, $GA(\mathcal{S})$) to denote the class of all A-infs (resp., wA-infs, GA-infs). By the above discussion, we have

Corollary 5.9. *$A(\mathcal{S})$ is a proper subclass of $wA(\mathcal{S}) \cap GA(\mathcal{S})$.*

We sum up briefly relationships among the above mentioned special classes of continuous information systems by the following diagram:



Relationships among special classes of continuous information systems

The following property of induced information systems will be used in the sequel.

Theorem 5.10. *Let D be a domain with a basis B and $\mathcal{S}(D, B)$ the information structure introduced in Definition 3.6. Then $\mathcal{S}(D, B)$ satisfies the following mixed transition condition:*

$$\text{(MT)} \qquad (\forall X, Y \in Con, \forall a \in A)(X \models Y \wedge Y \vdash a \Rightarrow X \vdash a).$$

6 Categorical aspects

We study relationships of wA-infs and domains from categorical aspects.

Definition 6.1. [16, 26] An approximable mapping

$f : (A, Con_A, \vdash_A) \to (B, Con_B, \vdash_B)$

between C-infs $(A, Con_A, \vdash_A)$ and $(B, Con_B, \vdash_B)$ is a relation $f \subseteq Con_A \times B$ satisfying the next 5 conditions:

(1) $((X f F) \wedge \emptyset \neq F \subseteq_{\text{fin}} B) \Rightarrow (\exists Z \in Con_B)(F \subseteq Z \wedge X f Z)$,

(2) $(X f Y \wedge Y \vdash_B b) \Rightarrow X f b$,

(3) $(X \vdash_A X' \wedge X' f b) \Rightarrow X f b$,

(4) $(X \subseteq X' \in Con_A \wedge X f b) \Rightarrow X' f b$,

(5) $(X f b) \Rightarrow (\exists X' \in Con_A)(\exists Y \in Con_B)(X \vdash_A X' \wedge X' f Y \wedge Y \vdash_B b)$,

where $X f Y$ means that $X f c$ for all $c \in Y$.

The composition $g \circ f \subseteq Con_A \times C$ of relations $f \subseteq Con_A \times B$ and $g \subseteq Con_B \times C$ is defined by

$$X(g \circ f)c \Leftrightarrow (\exists Y \in Con_B)(XfY \wedge Ygc),$$

for all $X \in Con_A$ and $c \in C$.

It is easy to check that the entailment relation $\vdash$ in a C-inf $\mathcal{S} = (A, Con, \vdash)$ is the identity approximable mapping
$Id_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ such that $X(Id_{\mathcal{S}})a$ if and only if $X \vdash a$ for all $X \in Con$ and $a \in A$.

- Let **CINF** (resp., **AINF**, **GAINF**, **WAINF**) be the category of

—— C-infs (resp., A-infs, GA-infs, wA-infs);

—— approximable mappings.

- Let **DOM** (resp., **ADOM**) be the category of

—— domains (resp., algebraic domains);

—— Scott continuous functions.

Proposition 6.2. [16] *Categories* **AINF**, **GAINF** *and* **ADOM** *are equivalent.*

In terms of abstract bases, Spreen and Xu showed in 2008 that

Lemma 6.3. [16, Coroally 5.1] *Categories* **CINF** *and* **DOM** *are equivalent.*

To prove category **WAINF** is equivalent to category **DOM**, next we give an outline to directly construct an equivalence of categories **CINF** and **DOM**. With this construction, one can easily see that, as a corollary, category **WAINF** is equivalent to category **DOM**.

Proposition 6.4. *Let* $f : (A, Con_A, \vdash_A) \to (B, Con_B, \vdash_B)$ *be an approximable mapping between C-infs* $\mathcal{S}_A = (A, Con_A, \vdash_A)$ *and* $\mathcal{S}_B = (B, Con_B, \vdash_B)$*. Then* $|f| : |\mathcal{S}_A| \to |\mathcal{S}_B|$ *defined by* $|f|(x) = \{b \in B : (\exists X \in Con)(X \subseteq_{\text{fin}} x \wedge X f b)\}$ *for all* $x \in |\mathcal{S}_A|$ *is a Scott continuous function.*

Lemma 6.5. *Define* $|\cdot| : \mathbf{CINF} \to \mathbf{DOM}$ *such that* $\forall \mathcal{S} \in ob(\mathbf{CINF})$, $|\cdot|(\mathcal{S}) = |\mathcal{S}| \in ob(\mathbf{DOM})$ *and* $\forall f \in mor(\mathbf{CINF})$, $|\cdot|(f) = |f| \in mor(\mathbf{DOM})$*. Then* $|\cdot|$ *is a functor.*

It is a routing work to show following.

Proposition 6.6. *Let $f : D \to E$ be a Scott continuous function between domains D with basis B and E with basis B'. Then $\mathcal{S}(f) : \mathcal{S}(D, B) \to \mathcal{S}(E, B')$ defined by $X\mathcal{S}(f)b \Leftrightarrow b \ll f(\vee X)$ for all $X \in Con$ and $b \in B'$ is an approximable mapping.*

Lemma 6.7. *Define $\mathcal{S}(\cdot) : \mathbf{DOM} \to \mathbf{CINF}$ such that $\forall D \in ob(\mathbf{DOM})$, $\mathcal{S}(\cdot)(D) = \mathcal{S}(D, D) \in ob(\mathbf{CINF})$ and $\forall f \in mor(\mathbf{DOM})$, $\mathcal{S}(\cdot)(f) = \mathcal{S}(f) \in mor(\mathbf{CINF})$. Then $\mathcal{S}(\cdot)$ is a functor.*

Theorem 6.8. *There are natural isomorphisms* $\alpha : \mathbf{I}_{\mathbf{DOM}} \to |\cdot| \circ \mathcal{S}(\cdot)$ *and* $\beta : \mathbf{I}_{\mathbf{CINF}} \to \mathcal{S}(\cdot) \circ |\cdot|$. *Thus, categories* **CINF** *and* **DOM** *are equivalent.*

Functors $|\cdot|$ and $\mathcal{S}(\cdot)$ can be restricted to categories **WAINF** and **DOM**. So, we have

Corollary 6.9. *Categories* **WAINF** *and* **DOM** *are equivalent.*

Next we turn to consider relationships of a wA-inf $\mathcal{S}$ and its induced information structure $w\mathcal{S}$ in category **CINF**.

Recall that a *section-retraction pair* (f, g) (cf. [1]) in a category means two morphisms $f : A \to B$ and $g : B \to A$ such that $g \circ f = Id_A$. In this case, f is called a *section*, g is called a *retraction* and A is called a *retract* of B.

Theorem 6.10. *Let* $\mathcal{S} = (A, Con, \vdash)$ *be a wA-inf satisfying Condition (MT) and* $w\mathcal{S}$**=** $(A, Con, \models)$ *the induced A-inf by* $\mathcal{S}$*. Then for all* $X \in Con$ *and* $a \in A$*,*

$$(X, a) \in \vdash \circ [\models \circ (\vdash \circ \models)] \Leftrightarrow (X, a) \in \vdash,$$

where the first $\vdash$: $w\mathcal{S} \to \mathcal{S}$ *and* $\models \circ (\vdash \circ \models) : \mathcal{S} \to w\mathcal{S}$ *are approximable mappings.*

Consequently, $(\models \circ (\vdash \circ \models), \vdash)$ *is a section-retraction pair, and* $\mathcal{S}$ *is a retract of* $w\mathcal{S}$ *in category* **WAINF**.

By **WAINF≃DOM**, we have

Corollary 6.11. *If* $\mathcal{S} = (A, Con, \vdash)$ *is a wA-inf satisfying Condition (MT), then* $|\mathcal{S}|$ *is a retract of* $|w\mathcal{S}|$*, where* $w\mathcal{S} = (A, Con, \models)$ *is the induced A-inf by* $\mathcal{S}$*.*

Since $\mathcal{S}(D, B)$ is a wA-inf satisfying Condition (MT), we have $\mathcal{S}(D, B)$ is a retract of $w\mathcal{S}(D, B)$ in category **WAINF**,

and correspondingly

$|\mathcal{S}(D, B)|$ is a retract of $|w\mathcal{S}(D, B)|$ in category **DOM**.

7 Related topics

Here are some Related further topics.

- Find applications for the w-construction.
- Use the information systems in model checking.
- use general information structures to represents quasicontinuous domains.
- Give the counterpart of powerdomain constructions for information systems.

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Thank You!