On equivalence relations generated by Cauchy sequences in countable metric spaces

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Outline

Borel sets and Borel functions

Definition

Polish space: a separable, completely metrizable topological space.

Let X, Y be two Polish spaces.

Definition

$\mathbf{B}(X)$: *Borel sets* of X is the σ-algebra generated by the open sets of X.

Definition

A function $f : X \to Y$ is *Borel function* if $f^{-1}(U)$ is Borel for U open in Y.

Borel hierarchy

$$\mathbf{\Sigma}^0_1 = \text{open}, \quad \mathbf{\Pi}^0_1 = \text{closed};$$

$$\mathbf{\Sigma}^0_2 = F_\sigma, \quad \mathbf{\Pi}^0_2 = G_\delta;$$

for $1 \leq \alpha < \omega_1$,

$$\mathbf{\Sigma}^0_\alpha = \{\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_{\alpha_n}, \alpha_n < \alpha\};$$

$$\mathbf{\Pi}^0_\alpha = \text{ the complements of } \mathbf{\Sigma}^0_\alpha \text{ sets};$$

$$\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha.$$

Borel reducibility between equivalence relations

Let X, Y be Polish spaces and E, F equivalence relations on X, Y respectively.

Definition

$E \leq_B F$: There is a Borel function $\theta : X \to Y$ such that, for all $x, y \in X$,

$$xEy \iff \theta(x)F\theta(y).$$

$E \sim_B F$: $E \leq_B F$ and $F \leq_B E$;
$E <_B F$: $E \leq_B F$ but not $F \leq_B E$.

$\mathbf{\Sigma}^1_1$ sets and $\mathbf{\Pi}^1_1$ sets

Definition

Let X be a Polish space. A subset $A \subseteq X$ is **analytic** (or $\mathbf{\Sigma}^1_1$) if there is a Polish space Y and a closed subset $C \subseteq X \times Y$ such that

$$x \in A \iff \exists y \in Y((x, y) \in C).$$

A subset $A \subseteq X$ is **co-analytic** (or $\mathbf{\Pi}^1_1$) if $X \setminus A$ is $\mathbf{\Sigma}^1_1$.

Theorem (Suslin)

Let $A \subseteq X$. Then A is Borel iff it is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

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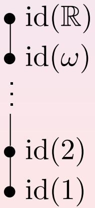
1st dichotomy theorem

We say an equivalence relation E on X is Borel, $\mathbf{\Sigma}^1_1$, or $\mathbf{\Pi}^1_1$ if $\{(x,y) \in X^2 : xEy\}$ is so in X^2.

Theorem (Silver, 1980)

Let E be a $\mathbf{\Pi}^1_1$ equivalence relation. Then

$$E \leq_B \mathrm{id}(\omega) \text{ or } \mathrm{id}(\mathbb{R}) \leq_B E.$$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2nd dichotomy theorem

Definition

E_0 is the equivalence relation on $\{0,1\}^\omega$ defined by

$$xE_0y \iff \exists m \forall n \geq m(x(n) = y(n)).$$

Fact: $E_0 \sim_B \mathbb{R}/\mathbb{Q}$.

Theorem (Harrington-Kechris-Louveau, 1990)

Let E be a Borel equivalence relation. Then either $E \leq_B \mathrm{id}(\mathbb{R})$ or $E_0 \leq_B E$.

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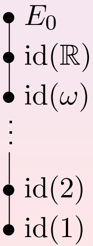
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3rd dichotomy theorem

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Fact: $E_1 = \mathbb{R}^\omega / c_{00}$, where $c_{00} = \bigcup_n \mathbb{R}^n$.

Theorem (Kechris-Louveau, 1997)

If $E \leq_B E_1$, then $E \leq_B E_0$ or $E \sim_B E_1$.

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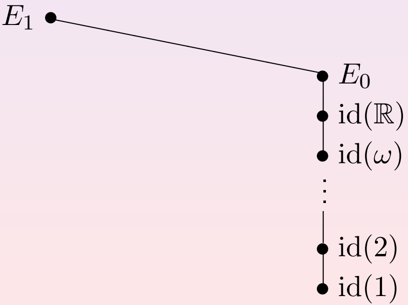
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4th dichotomy theorem

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Let E be an equivalence relation on X. The equivalence relation E^ω on X^ω defined by

$$xE^\omega y \iff \forall n(x(n)Ey(n)).$$

Fact: $E_0^\omega \sim_B \mathbb{R}^\omega/\mathbb{Q}^\omega$.

Theorem (Hjorth-Kechris, 1997)

If $E \leq_B E_0^\omega$, then $E \leq_B E_0$ or $E \sim_B E_0^\omega$.

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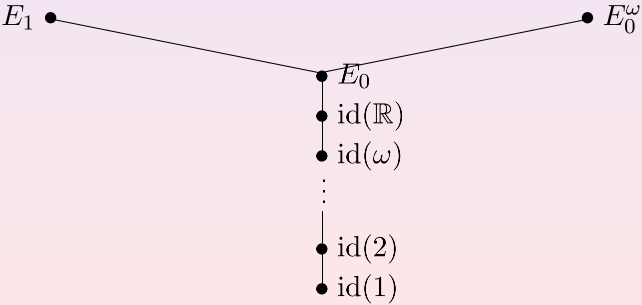
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Sequence equivalence relations

Definition

Let G be a Borel subgroup of $\mathbb{R}^\omega$, then the Borel equivalence relation $\mathbb{R}^\omega/G$ is defined by

$$x \text{ is equivalent to } y \iff x - y \in G.$$

Fact: $E_1 = \mathbb{R}^\omega/c_{00} = \mathbb{R}^\omega/\mathbb{R}^{<\omega}, \quad E_0^\omega \sim_B \mathbb{R}^\omega/\mathbb{Q}^\omega.$

Denote

$$c_0 = \{x \in \mathbb{R}^\omega : \lim_{n\to\infty} |x(n)| = 0\};$$

$$\ell_p = \{x \in \mathbb{R}^\omega : \sum_n |x(n)|^p < +\infty\};$$

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Theorem (Dougherty-Hjorth, 1999)

For $p, q \in [1+\infty)$, $p \leq q \iff \mathbb{R}^\omega/\ell_p \leq_B \mathbb{R}^\omega/\ell_q$.

Theorem (D. 2012)

For $p \in (0,1]$, *we have* $\mathbb{R}^\omega/\ell_p \sim_B \mathbb{R}^\omega/\ell_1$.

Theorem (Rosendal, 2005)

Every K_σ *equivalence relation on a Polish space is* $\leq_B \mathbb{R}^\omega/\ell_\infty$.

Corollary

E_1 *and* $\mathbb{R}^\omega/\ell_p$ *are* $\leq_B \mathbb{R}^\omega/\ell_\infty$.

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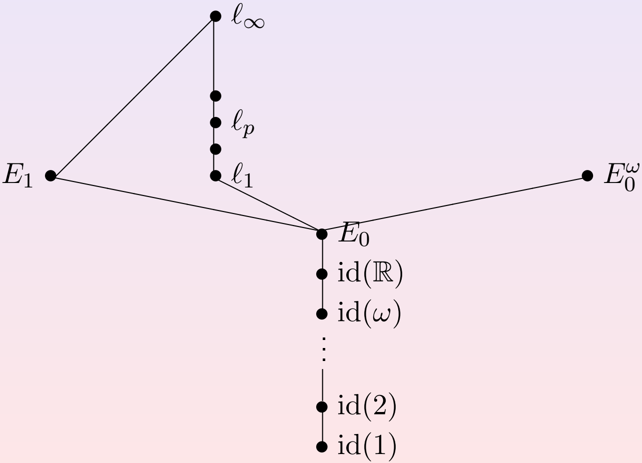
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For $p \in [1, +\infty)$, $\mathbb{R}^\omega/\ell_p$ and $\mathbb{R}^\omega/c_0$ are $\leq_B$ incomparable.

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$E_0^\omega \leq_B \mathbb{R}^\omega/c_0$.

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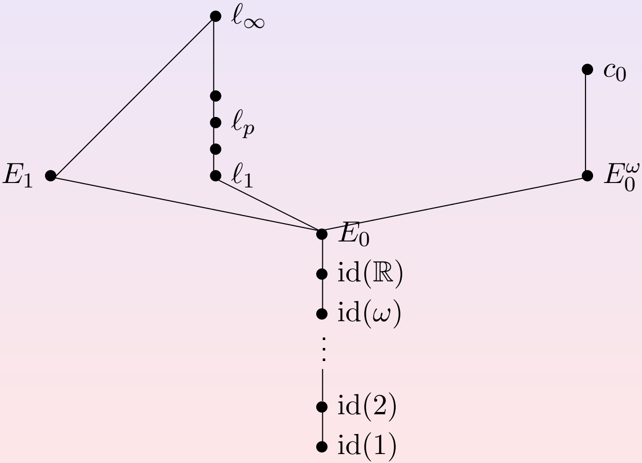
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$=^+$ is the equivalence relation on $\mathbb{R}^\omega$ defined by

$$x =^+ y \iff \{x(n) : n \in \omega\} = \{y(n) : n \in \omega\}.$$

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$E_0^\omega \leq_B =^+$, while $=^+$ and $\mathbb{R}^\omega / c_0$ are Borel incomparable.

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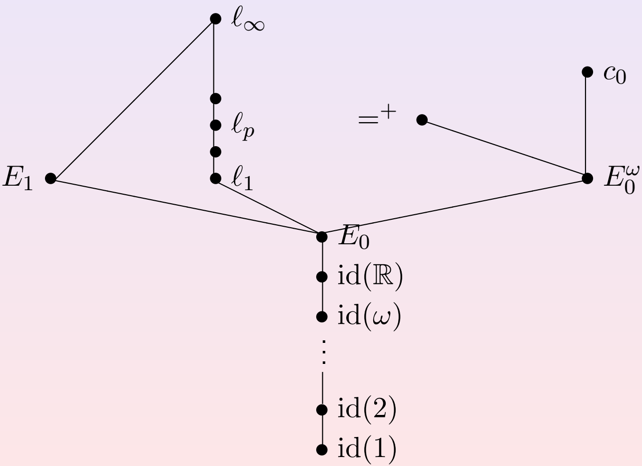
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Polish G-spaces and orbit equivalence relations

Definition

Polish group: A topological group whose underlying space is Polish.

G: Polish group,
X: Polish space,
$a : G \times X \to X$: continuous G-action on X.

Definition

Orbit equivalence relation:

$$xE_G^X y \iff \exists g \in G(g \cdot x = y).$$

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Σ^1_1 equivalence relations

Theorem (Kechris-Louveau, 1997)

$E_1 \not\leq_B E_G^X$ *for any Polish* G*-space* X.

E_0, E_1, $\mathbb{R}^\omega/\ell_p$, $\mathbb{R}^\omega/\ell_\infty$: F_σ equivalence relations;
E_0^ω, $\mathbb{R}^\omega/c_0$, $=^+$: $\mathbf{\Pi}^0_3$ equivalence relations.

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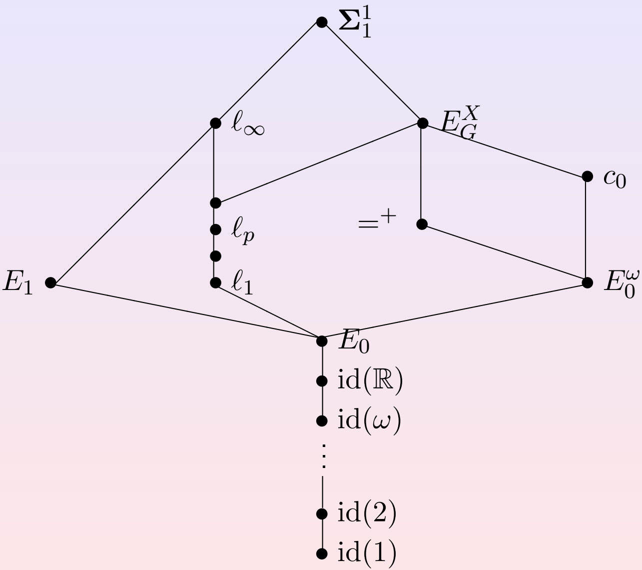
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Classification problems for Polish/compact metric spaces

Definition

Polish metric space: separable complete metric space.

1. $\mathrm{Iso}/\mathrm{Iso}_{\mathrm{cpt}}$: isometry among Polish/compact metric spaces
2. $\mathrm{Hom}/\mathrm{Hom}_{\mathrm{cpt}}$: homeomorphism ...
3. $\mathrm{Lip}/\mathrm{Lip}_{\mathrm{cpt}}$: Lipschitz isomorphism ...
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Note: $\mathrm{Uni}_{\mathrm{cpt}} = \mathrm{Hom}_{\mathrm{cpt}}$.

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Coding Polish metric spaces

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Let $\mathbb{X} \subseteq \mathbb{R}^{\omega\times\omega}$ consisting of elements $r = (r_{i,j})$ such that

(1) $\forall i,j \in \omega\,(r_{i,j} \geq 0 \wedge (r_{i,j} = 0 \iff i = j))$;

(2) $\forall i,j \in \omega\,(r_{i,j} = r_{j,i})$;

(3) $\forall i,j,k \in \omega\,(r_{i,j} \leq r_{i,k} + r_{j,k})$.

$\mathbb{X}$ is a Polish subspace of $\mathbb{R}^{\omega\times\omega}$.
Denote $\overline{X}_r$ the completion of (ω, r).

Definition

$\mathbb{X}_{\rm cpt} = \{r \in \mathbb{X} : \overline{X}_r \text{ is compact}\}$.

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Isometry and Homeomorpism

Theorem (Gromov)

$\mathrm{Iso}_{\mathrm{cpt}} \sim_B \mathrm{id}(\mathbb{R})$.

Theorem (Gao-Kechris, 2003)

Iso *is a universal orbit equivalence relation.*

Theorem (Zielinski, 2016)

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Hom *is an* Σ^1_2 *equivalence relation on* $\mathbb{X}$.

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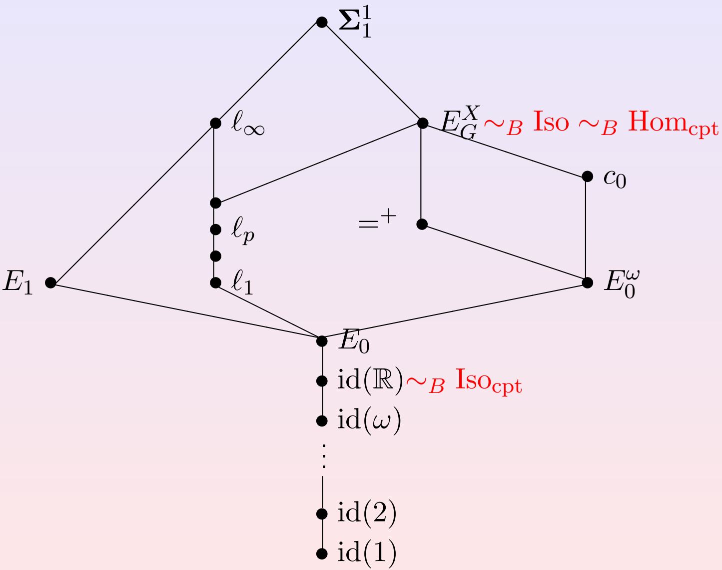
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Lipschitz isomorphism and uniform homeomorphsim

Theorem (Rosendal, 2005)

$\mathrm{Lip}_{\mathrm{cpt}} \sim_B \mathbb{R}^\omega/\ell_\infty$.

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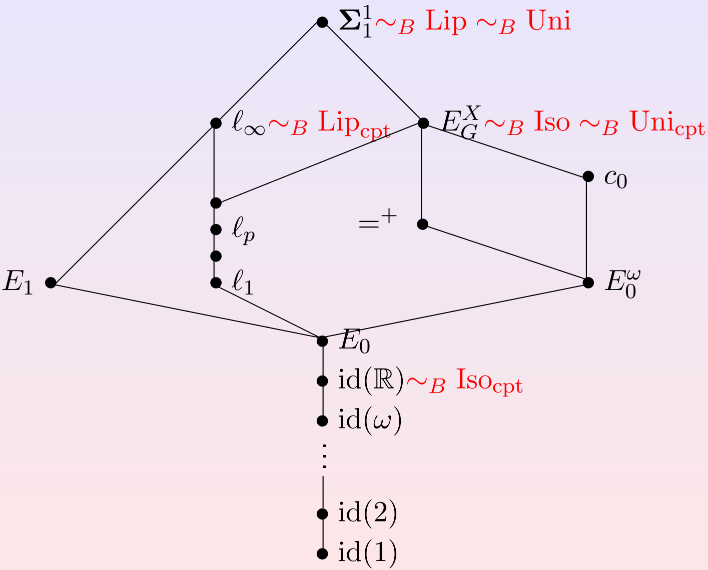
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Outline

Cauchy sequence equivalence relation

Fact

Let $r, s \in \mathbb{X}$. Then the following are equivalent:

(a) (ω, r) *and* (ω, s) *have the same set of Cauchy sequences;*

(b) *there exists a homeomorphism* $\varphi : \overline{X}_r \to \overline{X}_s$ *with* $\varphi \restriction \omega = \mathrm{id}(\omega)$.

Definition

Cauchy sequence equivalence relation: For $r, s \in \mathbb{X}$, $r E_{\mathrm{cs}} s$ iff (ω, r) and (ω, s) have the same set of Cauchy sequences.

Theorem (D.-Gu, 2018)

E_{cs} *is a* $\mathbf{\Pi}^1_1$*-complete equivalence relation. So* E_{cs} *and* Lip *(or* Uni*) are Borel incomparable.*

Cauchy sequence equivalence relation

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(a) *(ω, r) and (ω, s) have the same set of Cauchy sequences;*

(b) *there exists a homeomorphism $\varphi : \overline{X}_r \to \overline{X}_s$ with $\varphi \restriction \omega = \mathrm{id}(\omega)$.*

Definition

Cauchy sequence equivalence relation: For $r, s \in \mathbb{X}$, $r E_{\mathrm{cs}} s$ iff (ω, r) and (ω, s) have the same set of Cauchy sequences.

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Restriction on compact metric spaces

Denote $E_{\mathrm{csc}} = E_{\mathrm{cs}} \restriction \mathbb{X}_{\mathrm{cpt}}$.

Theorem (D.-Gu, 2018)

1. E_{csc} *is* $\mathbf{\Pi}^0_3$*-equivalence relation;*
2. $E_{\mathrm{csc}} \sim E_G^X$ *for some Polish group* G *and Polish* G*-space* X*;*
3. $\mathbb{R}^\omega / c_0 \leq_B E_{\mathrm{csc}}$;
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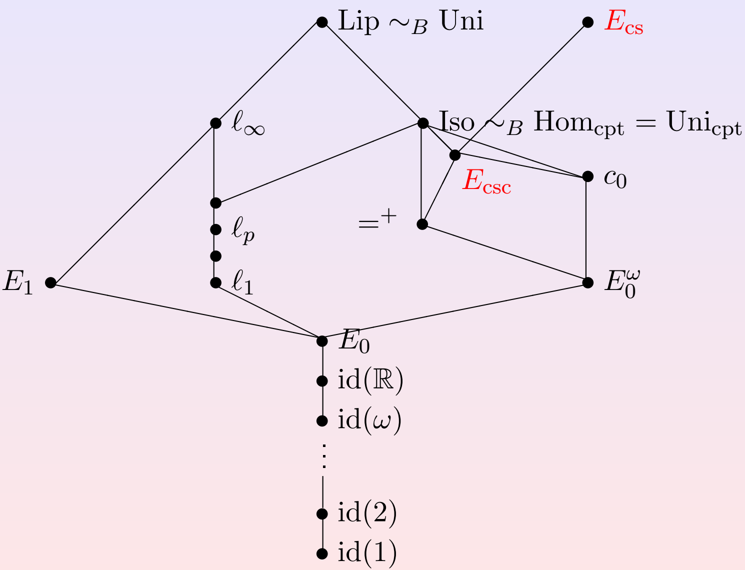
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ℓ_∞
Iso $\sim_B$ $\text{Hom}_{\text{cpt}} = \text{Uni}_{\text{cpt}}$
E_{csc}
c_0
$=^+$
ℓ_p
E_1
ℓ_1
E_0^ω
E_0
id($\mathbb{R}$)
id(ω)
id(2)
id(1)

Some invariant subsets of E_{csc}

$$\mathbb{X}_n = \{r \in \mathbb{X}_{\mathrm{cpt}} : \mathrm{card}(\overline{X}_r') = n\},$$

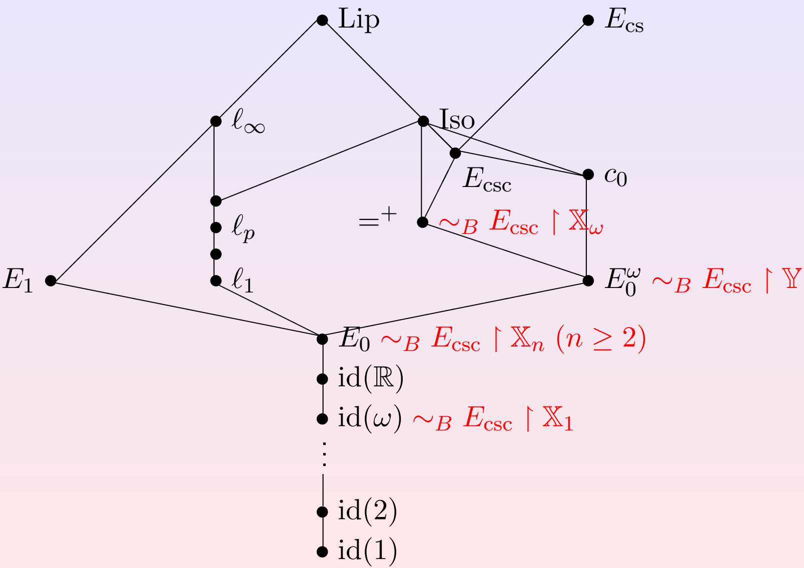
$$\mathbb{X}_\omega = \{r \in \mathbb{X}_{\mathrm{cpt}} : \mathrm{card}(\overline{X}_r'') = 1\}.$$

Fact

$$r \in \mathbb{X}_n \iff \overline{X}_r \cong \omega \cdot n + 1,$$

$$r \in \mathbb{X}_\omega \iff \overline{X}_r \cong \omega^2 + 1.$$

$$\mathbb{Y} = \{r \in \mathbb{X}_\omega : \overline{X}_r = \omega\}.$$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E_0^ω $\sim_B E_{csc} \upharpoonright \mathbb{Y}$
$E_0 \sim_B E_{csc} \upharpoonright \mathbb{X}_n$ $(n \geq 2)$
$\mathrm{id}(\mathbb{R})$
$\mathrm{id}(\omega) \sim_B E_{csc} \upharpoonright \mathbb{X}_1$
$\mathrm{id}(2)$
$\mathrm{id}(1)$

The end

Thank you!