Pmax-style extension for basis problem of uncountable linear order

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Basis problem for uncountable linear order

The basis problem for a certain class of mathematical structure focus on isolating the key structures and reducing the study of the whole structure to these key objects.

> *An analysis of a given class S of structures in this area frequently splits into two natural parts One part consists in recognizing the critical members of S while the other is in showing that a given list of critical members is in some sense complete. $\cdots$ To show that a given list S_0 of critical objects is exhaustive one needs to relate a given structure from S to one from the list S_0.* (Stevo Todorcevic, ICM 1998)

Some example of basis problems:

- (Day-Von Neumann) Does every non-amenable group contains F_2 as subgroup?
- (Subspace of Banach space) Does every Banach space contains l_p or c_0 as subspace?
- (Monster group) Classify the family of simple groups.
- (Maharams problem) If a submeasure is exhaustive, is it absolutely continuous with respect to a measure?

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Basis problems in set theory:

- (Subspace of regular spaces) Can the class of uncountable regular spaces has a 3-element basis consisting of $D(\omega_1)$, X and $X \times \{0\}$ where X is some uncountable subset of the unit interval and where $X \times \{0\}$ is considered as a subspace of the split-interval.
- (Basis of uncountable linear ordering) The class of all uncountable linear orderings has a 5-element basis X, ω_1, ω_1^*, C, C^* where X is some uncountable set of reals and where C is a countryman line.

Some examples of uncountable linear orders:

- uncountable subsets of real line(uncountable separable suborders)
- ω_1
- ω_1^*
- Suslin line

Notation: Those uncountable linear orders which do not contain uncountable separable suborders or copies of ω_1 or ω_1^* are called Aronszajn lines.

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Notation: Those uncountable linear orders which do not contain uncountable separable suborders or copies of ω_1 or ω_1^* are called Aronszajn lines. Suslin line is a Aronszajn line. The existence of Suslin line is independent of ZFC. However, Shelah is able to construct a special type of Aronszajn line under ZFC.

Theorem (Shelah)

There is a Countryman line.

Here a linear order $(X, <)$ is called Countryman if X^2 (viewed as a partial order) can be decomposed as ω many chain.

Minimal size of basis for linear order

Theorem (Baumgarterner)

Under PFA, all ω_1-dense suborder of real lines are isomorphic.

Theorem (Moore)

Under PFA, for any Countryman line C, all Aronszajn line contains C or C^* .

In conclusion, under PFA, there is a 5-element basis of uncountable linear order. To the contrast, the minimal size of basis could also be huge.

Theorem (Sierpinski)

Under CH, there is no basis for the uncountable separable linear orders of cardinality less than 2^{ω_1}.

Manipulating the size of basis of Countryman line

Theorem (Peng)

For any $n \leq \omega_1$, it is consistent that there is a 2^n-element basis for Countryman line.

How about Aronszajn line? The main obstacle is as follows: Peng's construction involves a special subclass of proper forcing poset Γ. However,

- It is unclear whether the poset used in Moore's proof are also in Γ.
- PFA requires countable support iteration, while it is unknown whether Γ is preserved under countable support iteration.

Peng's construction uses Aspero-Mota iteration, which only allows a relatively small subclass of poset.

$\mathbb{P}_{max}$

$\mathbb{P}_{max}$ is a forcing poset defined by Woodin in presence of determinacy assumption. A key feature of $\mathbb{P}_{max}$ theory is generic maximality. It captures the maximal Π_2-theory of $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ in the following sense. In a P_{max} extension V, for any Π_2 sentence ψ, if in some generic extension $V[G]$,

$$\langle H(\omega_2), \in, NS_{\omega_1}\rangle^{V[G]} \models \psi,$$

then in V, already we have

$$\langle H(\omega_2), \in, NS_{\omega_1}\rangle \models \psi.$$

As a consequence, $\mathbb{P}_{max}$ forced that there is a 5-element basis of uncountable linear order.

$\mathbb{P}_{max}$ variant

Woodin introduced a machinery for constructing variant of original $\mathbb{P}_{max}$ forcing. Following are some variant of $\mathbb{P}_{max}$:

- $\mathbb{P}_{max}$ for ω_1-density of NS_{ω_1}.
- $\mathbb{S}_{max}$ for a Suslin tree.
- $\mathbb{P}^{\clubsuit}$ for the $\clubsuit$ principle.

For a Σ_2 sentence ψ, if a variant $\mathbb{P}^{\psi}_{max}$ can be defined, then there is a Π^2-maximality theory for $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ conditional on ψ.

The forcing class Γ

During the construction, we will fix a tree $T \subset 2^{<\omega_1}$ and disjoint subsets $(X_i \mid i < n)$ of ω_1 with some prescribed nice properties. Let

$$P_{X_i} = \{p \in [T]^{<\omega} \mid \Delta(p) = \{\Delta(s,t) \mid s \perp t \text{ in p}\} \subset X_i\}$$

ordered by inclusion. P_{X_i} is proper.

Γ: all poset Q such that for all i, $Q \times P_{X_i}$ is proper.

Meanwhile, no good iteration theory for Γ is known. In particular, it is unknown how to force $BPFA_\Gamma$ using iterated forcing. On the other hand, the $\mathbb{P}_{max}$ machinery relies on iterated ultrapower rather than iterated forcing.

$\mathbb{P}_{max}$ variant for Γ

Following Woodin's generalized version of $\mathbb{P}_{max}$ forcing, we define the variant $\mathbb{P}^{\Gamma}_{max}$ as follows: The partial order $\mathbb{P}^{\Gamma}_{max}$ consists of all pairs $\langle (M, I), t, x, K\rangle$ such that

1. M is a countable transitive model of ZFC^-,
2. $I \in M$ and in M, I is a normal ideal on ω_1,
3. (M, I) is iterable,
4. M think t is a subtree of $2^{<\omega_1}$ and $x = (x_i \mid i < n)$ is a sequence of subset of ω_1, t and x has some prescribed nice properties, in particular, if we define P_x in M, then P_x is proper.
5. $K \in M$ and K is a set of pairs $(\langle (N, J), b, y, E\rangle, j)$ such that
    - $\langle (N, J), b, y, E\rangle \in \mathbb{P}^{\Gamma}_{max} \cap H(\omega_1)^M$,
    - j is an iteration of (N, J) of length ω_1^M such that $j(J) = I \cap j(N)$ and $j(b) = t$, $j(y) = x$
    - $j(E) \subset K$,

   with the property that for each $p \in \mathbb{P}^{\Gamma}_{max}$ there is at most one j such that $(p, j) \in X$.

Say $\langle (M', I'), t', x', K'\rangle < \langle (M, I), t, x, K\rangle$ if there exists a j such that $(\langle (M, I), t, x, K\rangle, j) \in K'$.

Proposition

$\mathbb{P}^{\Gamma}_{max}$ forces the following:

- *NS_{ω_1} is saturated, or $Sat(NS_{\omega_1})$ holds.*
- *All ω_1-dense subset of reals are isomorphic.*
- *There is a 2^n size basis for countryman line.*
- *If ground model satisfies $V = L(P(\mathbb{R})) + AD_{\mathbb{R}}$, then $BPFA_{\Gamma}$ holds.*

Konig-Moore-Velickovic shows that $Sat(NS_{\omega_1}) + BPFA$ implies every Aronzajn line contains a Countryman line. By analysing the forcing poset used in their proof, $Sat(NS_{\omega_1}) + BPFA_{\Gamma}$ also works for the purpose. Putting these together, we have

Theorem

Assuming $V = L(P(\mathbb{R})) + AD_{\mathbb{R}}$, $\mathbb{P}^{\Gamma}_{max}$ forces the minimal size of basis for uncountable linear order is $2^n + 3$.

Question

As a natural question, we are curious about basis of arbitrary size.

Question

For any fixed $n \geq 5$, is it consistent that the minimal size of basis for uncountable linear order is n.