The Brouwer Invariance Theorems in Reverse Mathematics

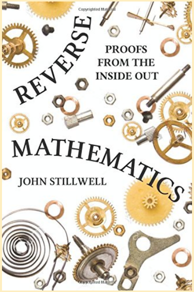
Takayuki Kihara[1]

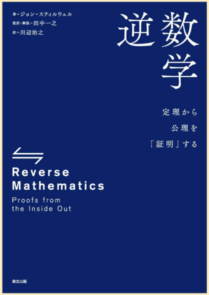
Nagoya University, Japan

The 9th international conference on Computability Theory and Foundations of Mathematics, Wuhan, China, March 24, 2019

[1] The speaker's research was partially supported by JSPS KAKENHI Grant 17H06738, 15H03634, the JSPS Core-to-Core Program (A. Advanced Research Networks), and the Young Scholars Overseas Visit Program in Nagoya University

Stillwell (2018) “Reverse mathematics”





- (Left) John Stillwell, Reverse mathematics. Proofs from the inside out. Princeton University Press, Princeton, NJ, 2018.
- (Right) Japanese translation (2019) by H. Kawabe and K. Tanaka.

A few months ago, Prof. Tanaka sent me a draft of the Japanese translation of John Stillwell's book, "*Reverse mathematics. Proofs from the inside out*".

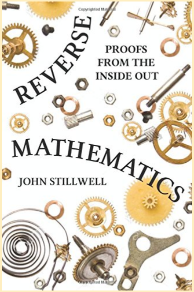
A few months ago, Prof. Tanaka sent me a draft of the Japanese translation of John Stillwell's book, "*Reverse mathematics. Proofs from the inside out*".

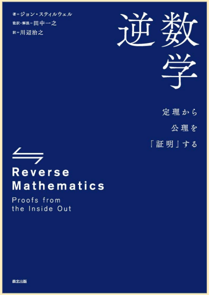
Then, I found the following paragraph:

しかしながら，（少なくとも 2 次元以上では）これらの不変性定理が RCA_0 で証明可能なのかはまだわかっていない．また，これらの定理が弱ケーニヒの補題を含意するかどうか，そしてその結果，弱ケーニヒの補題と同値かどうかもわかっていない．ブラウワーの不変性定理の正確な強さを把握することは，逆数学におけるもっとも興味深い未解決問題の一つだろう．

"Finding the exact strength of the Brouwer invariance theorems seems to me one of the most interesting open problems in reverse mathematics."
(Page 148 in Stillwell "Reverse Mathematics")

Stillwell (2018) “Reverse mathematics”





- (Left) John Stillwell, Reverse mathematics. Proofs from the inside out. Princeton University Press, Princeton, NJ, 2018.
- (Right) Japanese translation (2019) by H. Kawabe and K. Tanaka.

“Finding the exact strength of the Brouwer invariance theorems seems to me one of the most interesting open problems in reverse mathematics.”
(Page 148 in Stillwell “Reverse Mathematics”)

What are ... the Brouwer invariance theorems?

- (Cantor 1877) There is a bijection between $\mathbb{R}^m$ and $\mathbb{R}^n$.
- (Peano 1890) There is a continuous surjection from $\mathbb{R}^1$ onto $\mathbb{R}^n$.

The "invariance of dimension" problem

If $m \neq n$, prove that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

What are ... the Brouwer invariance theorems?

- (Cantor 1877) There is a bijection between $\mathbb{R}^m$ and $\mathbb{R}^n$.
- (Peano 1890) There is a continuous surjection from $\mathbb{R}^1$ onto $\mathbb{R}^n$.

The "invariance of dimension" problem

If $m \neq n$, prove that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

- Lüroth (1878) proved the invariance of dimension theorem for $n < m \leq 3$.

What are ... the Brouwer invariance theorems?

- (Cantor 1877) There is a bijection between $\mathbb{R}^m$ and $\mathbb{R}^n$.
- (Peano 1890) There is a continuous surjection from $\mathbb{R}^1$ onto $\mathbb{R}^n$.

The "invariance of dimension" problem

If $m \neq n$, prove that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

- Lüroth (1878) proved the invariance of dimension theorem for $n < m \leq 3$.
- Thomae (1878) announced the inv. of dim. theorem
- Netto (1879) announced the inv. of dim. theorem
- Cantor (1879) announced the inv. of dim. theorem

What are ... the Brouwer invariance theorems?

- (Cantor 1877) There is a bijection between $\mathbb{R}^m$ and $\mathbb{R}^n$.
- (Peano 1890) There is a continuous surjection from $\mathbb{R}^1$ onto $\mathbb{R}^n$.

The "invariance of dimension" problem

If $m \neq n$, prove that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

- Lüroth (1878) proved the invariance of dimension theorem for $n < m \leq 3$.
- Thomae (1878) announced the inv. of dim. theorem with an incorrect proof.
- Netto (1879) announced the inv. of dim. theorem with an incorrect proof.
- Cantor (1879) announced the inv. of dim. theorem with an incorrect proof.

What are ... the Brouwer invariance theorems?

- (Cantor 1877) There is a bijection between $\mathbb{R}^m$ and $\mathbb{R}^n$.
- (Peano 1890) There is a continuous surjection from $\mathbb{R}^1$ onto $\mathbb{R}^n$.

The "invariance of dimension" problem

If $m \neq n$, prove that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

- Lüroth (1878) proved the invariance of dimension theorem for $n < m \leq 3$.
- Thomae (1878) announced the inv. of dim. theorem with an incorrect proof.
- Netto (1879) announced the inv. of dim. theorem with an incorrect proof.
- Cantor (1879) announced the inv. of dim. theorem with an incorrect proof.
- During 1880s and 1890s, most mathematicians believed that the invariance of dimension problem had been solved (by Cantor and Netto).
- Jügens (1899) gave a critical account of the state of the problem.
- Shönflies (1899) claimed that the inv. of dim. problem is still open.

What are ... the Brouwer invariance theorems?

- (Cantor 1877) There is a bijection between $\mathbb{R}^m$ and $\mathbb{R}^n$.
- (Peano 1890) There is a continuous surjection from $\mathbb{R}^1$ onto $\mathbb{R}^n$.

The "invariance of dimension" problem

If $m \neq n$, prove that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

- Lüroth (1878) proved the invariance of dimension theorem for $n < m \leq 3$.
- Thomae (1878) announced the inv. of dim. theorem with an incorrect proof.
- Netto (1879) announced the inv. of dim. theorem with an incorrect proof.
- Cantor (1879) announced the inv. of dim. theorem with an incorrect proof.
- During 1880s and 1890s, most mathematicians believed that the invariance of dimension problem had been solved (by Cantor and Netto).
- Jügens (1899) gave a critical account of the state of the problem.
- Shönflies (1899) claimed that the inv. of dim. problem is still open.
- Lüroth (1899) announced the invariance of dimension theorem for $n < m \leq 4$ with an "extremely complicated proof".

What are ... the Brouwer invariance theorems?

Brouwer (1911) proved the following theorems:

1. *The Brouwer fixed point theorem*
2. *The no-retraction theorem*: The n-dimensional sphere is not a retract of the $(n+1)$-dimensional ball.
3. *The invariance of dimension theorem*: If $m < n$ then there is no continuous injection from $\mathbb{R}^n$ into $\mathbb{R}^m$
4. *The invariance of domain theorem*: Let $U \subseteq \mathbb{R}^m$ be an open set, and $f : U \to \mathbb{R}^m$ be a continuous injection. Then, the image $f[U]$ is also open.

- (Baire, Hadamard, Lebesgue) The invariance of domain theorem implies the invariance of dimension theorem.
- The invariance of domain theorem is used to show various important results, in particular, on topological manifolds.

What are ... the Brouwer invariance theorems?

Alexander duality $\Longrightarrow$ the Jordan-Brouwer separation theorem $\Longrightarrow$ invariance of domain $\Longrightarrow$ invariance of dimension

- *Alexander duality*: $\tilde{H}_q(E) \simeq \tilde{H}^{n-q-1}(\mathbb{S}^n \setminus E)$, where $\tilde{H}$ stands for reduced homology or reduced cohomology.
- *The Jordan-Brouwer separation theorem*: Let S^r be a homeomorphic copy of the r-sphere $\mathbb{S}^r$ in $\mathbb{S}^n$, then

$$\tilde{H}_q(\mathbb{S}^n \setminus S^r) \simeq \begin{cases} \mathbb{Z} & \text{if } q = n - r - 1 \\ 0 & \text{otherwise} \end{cases}$$

In particular, S^{n-1} separates $\mathbb{S}^n$ into two components, and these components have the same homology groups as a point. Moreover, S^{n-1} is the common boundary of these components.

In constructive mathematics

What axioms are needed to prove the Brouwer invariance theorems?

In constructive mathematics

What axioms are needed to prove the Brouwer invariance theorems?

- Orevkov (1963,1964): The no-retraction theorem and the Brouwer fixed-point theorem are false in the (Markov-style) constructive mathematics.

In constructive mathematics

What axioms are needed to prove the Brouwer invariance theorems?

- Orevkov (1963,1964): The no-retraction theorem and the Brouwer fixed-point theorem are false in the (Markov-style) constructive mathematics.
- Beeson "Foundations of Constructive Mathematics" (1985) claimed (without proof) the "*uniformly continuous*" versions of the no-retraction theorem and the invariance of dimension theorem are provable in (Bishop-style) constructive mathematics.

In constructive mathematics

What axioms are needed to prove the Brouwer invariance theorems?

- Orevkov (1963,1964): The no-retraction theorem and the Brouwer fixed-point theorem are false in the (Markov-style) constructive mathematics.
- Beeson “Foundations of Constructive Mathematics” (1985) claimed (without proof) the “*uniformly continuous*” versions of the no-retraction theorem and the invariance of dimension theorem are provable in (Bishop-style) constructive mathematics.
- Julian-Mines-Richman (1983) have studied the Alexander duality and the Jordan-Brouwer separation theorem in the context of Bishop-style constructive mathematics.

What is ... reverse mathematics?

What axioms are needed to prove the Brouwer invariance theorems?

- Reverse mathematics is a program to determine the exact (set-existence) axioms which are needed to prove theorems of ordinary mathematics.

What is ... reverse mathematics?

What axioms are needed to prove the Brouwer invariance theorems?

- Reverse mathematics is a program to determine the exact (set-existence) axioms which are needed to prove theorems of ordinary mathematics.
- We employ a subsystem **$\mathbf{RCA_0}$** of second order arithmetic as our base system, which consists of:
  1. Basic first-order arithmetic (e.g. the first-order theory of the non-negative parts of discretely ordered rings).
  2. Σ^0_1-induction schema.
  3. Δ^0_1-comprehension schema.
- Roughly speaking, **$\mathbf{RCA_0}$** corresponds to (non-uniform) computable mathematics (as Δ^0_1 = computable).

Some examples of reverse mathematics

The following are provable in RCA_0:

1. *Intermediate value theorem*.
2. *Urysohn's lemma*: Every separable metric space is perfectly normal.
3. *Tietze's extension theorem*: Every continuous function on a closed subset of a Polish space X into $[0, 1]$ can be extended to a continuous function on X into $[0, 1]$.
4. *Sperner's lemma* (a combinatorial analog of Brouwer's fixed point thm.)

The following are equivalent over RCA_0:

1. *Weak König's lemma*: Every infinite binary tree has an infinite path.
2. *The Heine–Borel theorem*: Every open cover of a totally bounded Polish space has a finite subcovering.
3. *The Jordan curve theorem*: The Jordan curve in $\mathbb{R}^2$ divides it into two open connected components.
4. *The Shönflies theorem*: Every Jordan curve is mapped onto the unit square by a homeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^2$.

Forward direction

WKL $\Longrightarrow$ Alexander duality $\Longrightarrow$ the Jordan-Brouwer separation $\Longrightarrow$ invariance of domain $\Longrightarrow$ invariance of dimension

Alexander duality: $\tilde{H}_q(E) \simeq \tilde{H}^{n-q-1}(\mathbb{S}^n \setminus E)$,
where $\tilde{H}$ stands for reduced homology or reduced cohomology.

homology theory in $\mathbf{WKL_0}$ (= $\mathbf{RCA_0}$+ weak König's lemma)

- We need $\mathbf{WKL_0}$ to proceed the barycentric subdivision argument.
- By barycentric subdivision, one can show the simplicial approximation theorem, which is needed to show basic facts on singular homology theory (alternatively, to show the topological invariance of simplicial homology).
- Similarly, $\mathbf{WKL_0}$ proves that these homology theories satisfy Eilenberg–Steenrod axioms, and so one can use the Mayer–Vietoris sequence.
- Hence, $\mathbf{WKL_0}$ proves (a spacial case of) the Alexander duality.

Note: Terence Tao (2014) gave a proof of the invariance of domain theorem without homology theory, which can also be carried out within $\mathbf{WKL_0}$.

Reverse direction

$\neg$**WKL** $\iff$ $\neg$ no-retraction theorem $\implies$ $\mathbf{S^1}$ is an absolute extensor $\implies$ $\mathbf{2}$-inessential $\implies$ $\mathbf{dim} \le \mathbf{1}$ $\implies$ embeddable into $\mathbb{R}^3$.

Fact (Orevkov 1963, Shioji-Tanaka 1990)

Over $\mathbf{RCA_0}$, the following are equivalent:

1. *Weak König's lemma*
2. *The Brouwer fixed point theorem*
3. *The no-retraction theorem*: The circle S^1 is not a retract of the disk.

Reverse direction

$\neg$**WKL** $\iff$ $\neg$ no-retraction theorem $\implies$ $\mathbf{S}^1$ is an absolute extensor $\implies$ $\mathbf{2}$-inessential $\implies$ $\mathbf{dim} \leq \mathbf{1}$ $\implies$ embeddable into $\mathbb{R}^3$.

A space K is called an *absolute extensor* for X if for any continuous map $f : P \to K$ on a closed set $P \subseteq X$, one can find a continuous map $g : X \to K$ extending f.

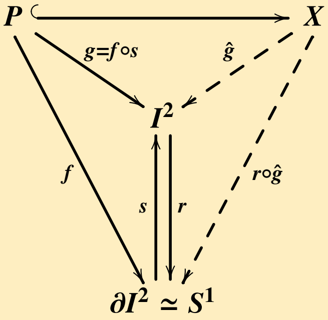
Tietze's extension theorem (RCA$_0$)

The n-hypercube I^n is an absolute extensor for any Polish space.

$\neg$**WKL** $\iff$ $\neg$ no-retraction theorem $\implies$ $\mathbf{S^1}$ is an absolute extensor $\implies$ $\mathbf{2}$-inessential $\implies$ $\mathbf{dim} \leq \mathbf{1}$ $\implies$ embeddable into $\mathbb{R}^3$.

Lemma ($\mathbf{RCA_0}$)

If the no-retraction theorem fails, then the $\mathbf{1}$-dimensional sphere S^1 is an absolute extensor for any Polish space.



$\neg$**WKL** $\iff$ $\neg$ no-retraction theorem $\implies$ $\mathbf{S}^1$ is an absolute extensor $\implies$ $\mathbf{2}$-inessential $\implies$ $\mathbf{dim} \leq \mathbf{1}$ $\implies$ embeddable into $\mathbb{R}^3$.

The notion of an absolute extensor plays a key role in topological dimension theory (e.g. Dranishnikov's extension dimension theory).

Fact (Eilenberg-Otto? Alexandroff?)

1. The covering dimension of X is $\leq n$ $\iff$ the n-sphere $\mathbf{S}^n$ is an absolute extensor for X.
2. The cohomological dimension of X (w.r.t. coefficient G) is $\leq n$ $\iff$ the Eilenberg-MacLane complex $K(G,n)$ is an absolute extensor for X.

$\neg$WKL $\iff$ $\neg$ no-retraction theorem $\implies$ $\mathbf{S}^1$ is an absolute extensor $\implies$ $\mathbf{2}$-inessential $\implies$ $\mathbf{dim} \leq \mathbf{1}$ $\implies$ embeddable into $\mathbb{R}^3$.

The notion of an absolute extensor plays a key role in topological dimension theory (e.g. Dranishnikov's extension dimension theory).

Fact (Eilenberg-Otto? Alexandroff?)

1. The covering dimension of X is $\leq n$ $\iff$ the n-sphere $\mathbf{S}^n$ is an absolute extensor for X.
2. The cohomological dimension of X (w.r.t. coefficient G) is $\leq n$ $\iff$ the Eilenberg-MacLane complex $K(G, n)$ is an absolute extensor for X.

- We have shown that if the no-retraction theorem fails, then the $\mathbf{1}$-sphere $\mathbf{S}^1$ is an absolute extensor for any Polish space.
- Classically, this means that: *every Polish space is at most one-dimensional*!

¬**WKL** $\iff$ ¬ no-retraction theorem $\implies$ $\mathbf{S}^1$ is an absolute extensor $\implies$ $\mathbf{2}$-inessential $\implies$ $\mathbf{dim} \leq \mathbf{1}$ $\implies$ embeddable into $\mathbb{R}^3$.

A sequence $(A_i, B_i)_{i \leq n}$ of disjoint pairs of closed sets in X is *inessential* if there is a sequece $(U_i, V_i)_{i \leq n}$ of disjoint open sets in X s.t.

- $A_i \subseteq U_i$ and $B_i \subseteq V_i$ for each $i \leq n$
- and $(U_i \cup V_i)_{i<n+1}$ covers X.

Lemma (RCA$_0$)

Let X be a Polish space. If the n-sphere $\mathbf{S}^n$ is an absolute extensor for X, then X has no essential sequence of length $n + 1$.

Indeed, one can show the "effective" version; that is, given $(A_i, B_i)_{i \leq n}$, one can effectively find such a $(U_i, V_i)_{i \leq n}$.
In this case, we say that X is *effectively* $(n + 1)$*-inessential*.

¬**WKL** $\iff$ ¬ no-retraction theorem $\Longrightarrow$ $\mathbf{S}^1$ is an absolute extensor $\Longrightarrow$ $\mathbf{2}$-inessential $\Longrightarrow$ $\mathbf{dim} \leq \mathbf{1}$ $\Longrightarrow$ embeddable into $\mathbb{R}^3$.

(Lebesgue) Let $\mathcal{U}$ be a cover of a space X.

- The order of $\mathcal{U}$ is $\leq n$ $\iff$ $\forall U_0, U_1, \ldots, U_{n+1} \in \mathcal{U}$ we have $\bigcap_{i<n+2} U_i = \emptyset$.
- The covering dimension of X is $\leq n$ $\iff$ for any finite open cover of X, one can effectively find a finite open refinement of order $\leq n$.

Fact (Eilenberg-Otto)

The covering dimension of X is at most n
$\iff$ X has no essential sequence of length $n + 1$.

Lemma (RCA$_0$)

A Polish space X is effectively $(n + 1)$-inessential
$\Longrightarrow$ the covering dimension of X is effectively at most n.

(Proof) Formalize the standard proof.

¬**WKL** $\iff$ ¬ no-retraction theorem $\implies$ $\mathbf{S}^1$ is an absolute extensor $\implies$ $\mathbf{2}$-inessential $\implies$ $\mathbf{dim \leq 1}$ $\implies$ embeddable into $\mathbb{R}^3$.

The Nöbeling imbedding theorem

If a separable metrizable space X is at most n-dimensional, then X can be topologically embedded into $\mathbb{R}^{2n+1}$.

- The nerve of a finite open cover $\mathcal{U} = (U_i)_{i<k}$ is a simplicial complex $N(\mathcal{U})$ with vertices $\{p_i\}_{i<k}$ such that an m-simplex $\{p_{j_0}, \dots, p_{j_{m+1}}\}$ belongs to $N(\mathcal{U})$ $\iff$ $U_{j_0} \cap \cdots \cap U_{j_{m+1}} = \emptyset$.
- The order of $\mathcal{U}$ is $\leq n$ $\implies$ one can give a geometric realization of the simplicial complex $N(\mathcal{U})$ in $\mathbb{R}^{2n+1}$ (by the so-called κ-mapping).

The Nöbeling imbedding theorem in $\mathbf{RCA_0}$

If a Polish space X is effectively at most n-dimensional, then X can be topologically embedded into $\mathbb{R}^{2n+1}$.

(Proof) Formalize the standard proof.

$\neg$**WKL** $\iff$ $\neg$ no-retraction theorem $\implies$ $\mathbf{S^1}$ is an absolute extensor $\implies$ $\mathbf{2}$-inessential $\implies$ $\mathbf{dim \leq 1}$ $\implies$ embeddable into $\mathbb{R}^3$.

Theorem (**RCA$_0$ + $\neg$WKL**)

- S^1 is a retract of the disk.
- S^1 is an absolute extensor for any Polish space.
- No Polish space has an essential sequence of length $\mathbf{2}$.
- The covering dimension of any Polish space is $\leq \mathbf{1}$.
- Every Polish space topologically embeds into $\mathbb{R}^3$.
- In particular, $\mathbb{R}^4$ topologically embeds into $\mathbb{R}^3$.
- Consequently, the invariance of dimension theorem fails.

Remark (Stillwell): **RCA$_0$** proves that $\mathbb{R}^2$ does not topologically embed into $\mathbb{R}$.

Theorem (K.)

The following are equivalent over $\mathbf{RCA_0}$:

1. *Weak König's lemma*
2. *The Brouwer fixed point theorem*
3. *The no-retraction theorem*: The n-dimensional sphere is not a retract of the $(n+1)$-dimensional ball.
4. *The invariance of dimension theorem*: If $m < n$ then there is no continuous injection from $\mathbb{R}^n$ into $\mathbb{R}^m$
5. *The invariance of domain theorem*: Let $U \subseteq \mathbb{R}^m$ be an open set, and $f : U \to \mathbb{R}^m$ be a continuous injection. Then, the image $f[U]$ is also open.

This solves Stillwell's problem.

Relationship with other works in computability theory

A space is *countable dimensional* if it is a countable union of $\mathbf{0}$-dim. subspaces.

Theorem (K.)

The following are equivalent over $\mathbf{RCA_0}$:

1. Weak König's lemma.
2. The Hilbert cube is not countable dimensional.

Proof

- (1)⇒(2): The usual argument only uses the Brouwer fixed point theorem, which can be carried out in $\mathbf{WKL_0}$.
- (2)⇒(1): If we assume $\neg\mathbf{WKL}$ then the Hilbert cube is one-dimensional, and therefore, it embeds into the one-dimensinal Nöbeling space, which is a finite union of zero dimensional subspaces.

A space is *countable dimensional* if it is a countable union of $\mathbf{0}$-dim. subspaces.

Theorem (K.)

The following are "instance-wise" equivalent over $\mathbf{RCA_0}$:

1. Weak König's lemma.
2. The Hilbert cube is not countable dimensional.

(Meta-reverse mathematics) The interpretation of the above theorem *in ω-models* is "equivalent" to the following theorem:

Theorem (J. Miller 2004)

1. If $\mathbf{a}$ and $\mathbf{b}$ are total degrees and $\mathbf{b} \ll \mathbf{a}$, then there is a non-total continuous degree $\mathbf{v}$ with $\mathbf{b} < \mathbf{v} < \mathbf{a}$.
2. If $\mathbf{v}$ is a non-total continuous degree and $\mathbf{b} < \mathbf{v}$ is total, then there is a total degree $\mathbf{c}$ with $\mathbf{b} \ll \mathbf{c} < \mathbf{v}$.

J. Miller's work on continuous degrees (2004)

Question (Pour-El and Lempp)

Does every $f \in C[0,1]$ have a code of least Turing degree?

Answer by J. Miller (2004)

No. There is $f \in C[0,1]$ with no easiest code w.r.t. Turing reducibility.

J. Miller's work on continuous degrees (2004)

Question (Pour-El and Lempp)

Does every $f \in C[0,1]$ have a code of least Turing degree?

Answer by J. Miller (2004)

No. There is $f \in C[0,1]$ with no easiest code w.r.t. Turing reducibility.

- The degree of difficulty of computing a code of $f \in C[0,1]$ is called the continuous degree of f.
- If f has a code of least Turing degree, then such a degree is called total.
- $a \ll b \;:\Longleftrightarrow\;$ every infinite binary tree $\leq_T a$ has a path $\leq_T b$.

Theorem (J. Miller 2004)

1. *Every PA-degree computes a counterexample to the question*:
   If $\mathbf{a}$ and $\mathbf{b}$ are total degrees and $\mathbf{b} \ll \mathbf{a}$, then there is a non-total continuous degree $\mathbf{v}$ with $\mathbf{b} < \mathbf{v} < \mathbf{a}$.
2. *Every counterexample yields a Scott set (an ω-model of* $\mathbf{WKL_0}$*)*:
   If $\mathbf{v}$ is a non-total continuous degree and $\mathbf{b} < \mathbf{v}$ is total, then there is a total degree $\mathbf{c}$ with $\mathbf{b} \ll \mathbf{c} < \mathbf{v}$.

WKL $\iff$ Hilbert cube is not countable dimensional.

An instance-wise interpretation in an ω-model $(\omega, \mathcal{S})$ of **RCA$_0$**:

- $\Rightarrow$ Let $(S_e)_{e\in\omega} \in \mathcal{S}$ be a sequence of copies of subspaces of ω^ω in I^ω, Then, there is an infinite binary tree $T \in \mathcal{S}$ satisfying the following: Every infinite path through T computes a point $x \in I^\omega$ such that x is not a point of S_e for any $e \in \omega$.
- $\Leftarrow$ Let $T \in \mathcal{S}$ be an infinite binary tree. Then, there is a sequence $(S_e)_{e\in\omega} \in \mathcal{S}$ of copies of subspaces of ω^ω such that, if $x \in I^\omega$ is not a point in S_e for any $e \in \omega$, then x computes an infinite path through T.

Theorem (J. Miller 2004)

1. If $\mathbf{a}$ and $\mathbf{b}$ are total degrees and $\mathbf{b} \ll \mathbf{a}$, then there is a non-total continuous degree $\mathbf{v}$ with $\mathbf{b} < \mathbf{v} < \mathbf{a}$.
2. If $\mathbf{v}$ is a non-total continuous degree and $\mathbf{b} < \mathbf{v}$ is total, then there is a total degree $\mathbf{c}$ with $\mathbf{b} \ll \mathbf{c} < \mathbf{v}$.

Theorem (K.)

The following are “instance-wise” equivalent over $\mathbf{RCA_0}$:

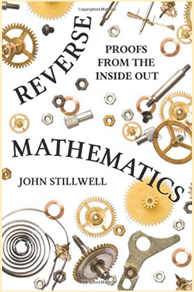
1. Weak König’s lemma.
2. The Hilbert cube is not countable dimensional.

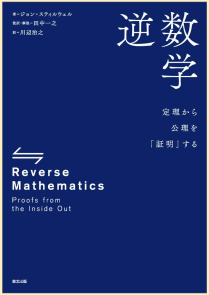
(Meta-reverse mathematics) The interpretation of the above theorem *in ω-models* is “equivalent” to the following theorem:

Theorem (J. Miller 2004)

1. If $\mathbf{a}$ and $\mathbf{b}$ are total degrees and $\mathbf{b} \ll \mathbf{a}$, then there is a non-total continuous degree $\mathbf{v}$ with $\mathbf{b} < \mathbf{v} < \mathbf{a}$.
2. If $\mathbf{v}$ is a non-total continuous degree and $\mathbf{b} < \mathbf{v}$ is total, then there is a total degree $\mathbf{c}$ with $\mathbf{b} \ll \mathbf{c} < \mathbf{v}$.

References





- (Left) John Stillwell, Reverse mathematics. Proofs from the inside out. Princeton University Press, Princeton, NJ, 2018.
- (Right) Japanese translation (2019) by H. Kawabe and K. Tanaka.

Thank you for your attention!