Working with computably Lipschitz reducibility above (uniformly) non-low_2 c.e. degrees

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Two non-low_2-ness notions

A Turing degree $\mathbf{d}$ is non-low_2 if for any total function $f \leq_T \emptyset'$ there is a total function $g \leq_T \mathbf{d}$ which is not dominated by f, i.e. , $\exists^\infty n[g(n) \geq f(n)]$.

A Turing degree $\mathbf{d}$ is uniformly non-low_2 if there is a computable function l such that if $\Phi_e^{\emptyset'}$ is total then $\Phi_{l(e)}^{\mathbf{d}}$ is total and not dominated by it.

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Proposition(Fan,2017)

There is an incomplete uniformly non-low_2 c.e. degree $\mathbf{d}$.

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There is a non-low_2 c.e. degree $\mathbf{d}$ which is not uniformly non-low_2.

A Turing degree $\mathbf{d}$ is array non-computable if for any total function $f \leq_{wtt} \emptyset'$ there is a total function $g \leq_T \mathbf{d}$ which is not dominated by f, i.e. $\exists^\infty n[f(n) \geq g(n)]$.

A Turing degree $\mathbf{d}$ is totally ω-c.e. if every total function $g \leq_T \mathbf{d}$ is ω-c.e..

In the c.e. Turing degrees,

$$\{\mathrm{uniformly\ non-low_2}\} \subsetneq \{\mathrm{non-low_2}\} \subsetneq$$

$$\{not\ \ totally\ \ \omega - c.e.\} \subsetneq \{array\ \ non-computable\}.$$

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Computable Lipchitz reducibility

Given two sequences like: " $\underbrace{11111\cdots111111}_{35\ consecutive\ numbers\ 1s}$ $\cdots$ and

$$\underbrace{10110101110101011110000101010001011}_{35\ \ numbers}\cdots."$$

- Let M be a Turing machine: $M(\tau) = \sigma = 2^{2^{2^{\tau}}}$. Then τ is an *M-description* of σ.
- For instance, if $\tau = 101$, then

$$M(\tau) = \sigma = 2^{2^{32}} = 2^{4294967296}$$

and $|\sigma| = 2^{32}$.

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- The Kolmogorov complexity of a string σ with respect to M via

$$C_M(\sigma) = min\{|\tau|, \infty \ : \ M(\tau) = \sigma\},$$

where $min\, \emptyset = \infty$.

- For a universal machine U, $C(\sigma) = C_U(\sigma) \leq C_M(\sigma) + O(1)$.

- In randomness and incomputability we have two fundamental measures: the plain complexity C and the prefix-free complexity K.

- Given M and U are prefix-free, $K_M(\sigma)$ and $K(\sigma) = K_U(\sigma)$ are well-defined.

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Definition (Levin 1974, Chaitin 1975)

A real α is 1-random if $\forall n[K(\alpha \upharpoonright n) > n - c]$.

Definition (Martin-Löf, P., 1966)

A real α is Martin-Löf random if for all computable collections of c.e. open sets $\{U_n : n \in \omega\}$, with $\mu(U_n) \leq 2^{-n}, \alpha \notin \cap_n U_n$.

Theorem (Schnorr, 1973)

The following are equivalent for a real α.

1. α is 1-random;
2. α is ML-random;.
3. no c.e. Martingale succeeds on it.

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- Real α is Δ^0_2 (left-c.e.) if it is the limit of a computable (increasing) sequence of rational numbers.
- For a universal prefix-free machine U, $\Omega_U = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}$ is a left-c.e. random real.
- $\alpha \leq_K \beta$ if $K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + O(1)$.
- $\alpha \leq_C \beta$ if $C(\alpha \upharpoonright n) \leq C(\beta \upharpoonright n) + O(1)$.
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Computable Lipschitz reducibility

Definition (Downey,Hirschfeldt,2008; Barmpalias and Lewis 2006)

Given two reals α and β, α is computable Lipschitz ($\leq_{cl}$) to β if there is a Turing functional Γ and a constant c such that $\alpha = \Gamma^{\beta}$ and the use of Γ on any argument n is bounded by $n + c$.

- (Soare,2013) The identity bound Turing reducibility (ibT).
- If $\alpha \leq_{cl} \beta$, then for all n, $K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + O(1)$.
- The cl-degree only contains either only random reals or non-random reals.
- (Downey, Hirschfeldt,Lafort 2001) The cl-degrees of left-c.e. reals is neither a lower semi-lattice, nor an upper semi-lattice.

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Computable Lipschitz reducibility

- (Downey, Hirschfeldt,Lafort 2001) There is no cl-complete left-c.e. real.
- (Yu and Ding,2004) There are two c.e.reals α and β which have no common upper bound under cl-reducibility in left-c.e. reals.
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Array non-computability and Computable Lipschitz reducibility

Theorem (Barmpalias, Downey and Greenberg,2010)

For a c.e. degree **d**, the following are equivalent:

(1) **d** is array non-computable.

(2) There is a cl-maximal pair of left-c.e. reals (α, β) in **d**.

(3) There is a left-c.e. real β in **d** which is not cl-reducible to any random left-c.e. real.

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- (α, β) is a cl-maximal pair of left-c.e. reals if no left-c.e. real can cl-compute both of them.
- (A, B) is a cl-maximal pair of c.e. sets if no c.e. set can cl-compute both of them.
- (Barmpalias,2005; Fan and Lu,2005) There exists a cl-maximal pair of c.e. sets.

Theorem (Ambos-spies, Ding, Fan and Wolfgang, 2013)

For a c.e. Turing degree $\mathbf{d}$, the following are equivalent:
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- (A, B) is a cl-maximal pair of c.e. sets if no c.e. set can cl-compute both of them.
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Theorem (Ambos-spies, Ding, Fan and Wolfgang, 2013)

For a c.e. Turing degree $\mathbf{d}$, the following are equivalent:
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Totally ω-c.e. degrees and Computable Lipschitz reducibility

- (Kjos-Hanssen, Wolfgang, Stephen, 2006) A set A is complex if there is an order (nondecreasing, unbounded, computable) function h such that $K(A \upharpoonright x) > h(x)$ for all x.
- (Downey, Hirschfeldt,2004) There is a real (not c.e.) which is not cl-reducible to any random real (indeed to any complex real).
- A degree $\mathbf{d}$ is called generalised low_2 if $\mathbf{d}'' \leq (\mathbf{d} \vee 0)'$.
- (Barmpalias, Downey, Greenberg, 2010) If $\mathbf{d}$ is not generalised low_2, then there is some $A \leq_T \mathbf{d}$ which is not ibT-reducible to any complex real.

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For a c.e. Turing degree $\mathbf{d}$, the following are equivalent:

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Theorem (Fan, 2017)

If a c.e. degree **d** is uniformly non-low_2, then

for any non-computable Δ^0_2 real α, there is a left-c.e. real β in **d** such that both of them have no common upper bound of c.e. reals under cl-reducibility.

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If a c.e. degree **d** is uniformly non-low_2, then

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There is no cl-maximal c.e set.

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- There exists a quasi-maximal cl-degree, i.e. there exits a real α, such that, for all reals β, if $\alpha \leq_{cl} \beta$, then $\beta \leq_T \alpha$. In fact, every random real satisfies the quasi-maximality property.
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- How to characterize in c.e. Turing degrees: uniformly non-low_2-ness or highness by cl-properties?
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