Ramsey property and infinite game in second-order arithmetic

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Abstract

This is an introductory talk about *Ramsey property* and *determinacy of infinite games*.
They are both the properties of sets of reals, i.e. subsets of $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$.

This is ongoing work (in progress) to find the relation between Ramseyness and determinacy within *second-order arithmetic*.

Outline:

1 Ramsey property

2 Determinacy of infinite games

3 Second-order arithmetic

4 Ramsey property and determinacy

Notation

- $2 = \{0, 1\}$
- $2^{<\mathbb{N}}$ $(= 2^*) := \bigcup_{n\in\mathbb{N}} 2^n$
  = (the set of finite sequences of 0 and 1)
- $2^{\mathbb{N}}$ = (the set of infinite sequences of 0 and 1)

$\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$ by identifing $\mathbb{N} \supseteq X = \chi_X \in 2^{\mathbb{N}}$

For $s = (s_0, \dots, s_{n-1}) \in 2^{<\mathbb{N}}$ and $x = (x_0, x_1, \dots) \in 2^{\mathbb{N}}$, write

$$s \subseteq x :\Leftrightarrow \forall i < n(s_i = x_i).$$

For $s \in 2^{<\mathbb{N}}$, put

$$[s] = \{ x \in 2^{\mathbb{N}} : s \subseteq x \}.$$

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Ramsey property

Definition (Ramsey property)

Given $P \subseteq 2^{\mathbb{N}}$, we say that P is *Ramsey* if either

$$\exists H \underset{\text{inf.}}{\subseteq} \mathbb{N}\ \forall X \underset{\text{inf.}}{\subseteq} H\ (X \in P) \quad \text{or} \quad \exists H \underset{\text{inf.}}{\subseteq} \mathbb{N}\ \forall X \underset{\text{inf.}}{\subseteq} H\ (X \notin P)$$

holds.

$P = 2^{\mathbb{N}}$ is Ramsey.

$\because$) Let $H = \mathbb{N}$. Then $\forall X \underset{\text{inf.}}{\subseteq} H\ (X \in P)$.

$P = [(1)] = \{ X \subseteq \mathbb{N} : 0 \in X \}$ is Ramsey.

$\because$) Let $H = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \ldots\}$. Then $\forall X \underset{\text{inf.}}{\subseteq} H\ (X \notin P)$.

On the other hand,

Axiom of Choice implies "$\exists P$ (P is not Ramsey)."

However, we can say P is Ramsey when P is simple enough.

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Motivation

For a set S,

$$[S]^n := \{ s \subseteq S : |s| = n \}$$

(the set of unordered n-tuples in S).

The *infinite Ramsey theorem* for n-tuples and 2-colors states that

$$\forall C : [\mathbb{N}]^n \to 2\ \exists H \underset{\text{inf.}}{\subseteq} \mathbb{N}\ (\forall x \in [H]^n\ C(x) = 0 \text{ or } \forall x \in [H]^n\ C(x) = 1),$$

while "every $P \subseteq 2^{\mathbb{N}}$ is Ramsey" is almost the same assertion as

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Open sets are Ramsey

Introduce the topology over $2^{\mathbb{N}}$ by taking $\{ [s] : s \in 2^{<\mathbb{N}} \}$ as open basis.
(This is the same topology as the product topology $2^{\mathbb{N}}$ where each 2 is discrete.)
The topological space $2^{\mathbb{N}}$ with this topology is called *Cantor space*.

Theorem

Every open set $P \subseteq 2^{\mathbb{N}}$ is Ramsey.

Proof: Later.

Ramseyness on each class

- Every open(Σ^0_1) set is Ramsey.
- Every Borel(Δ^1_1) set is Ramsey. [Galvin–Prikry '73]
- Every analytic(Σ^1_1) set is Ramsey. [Silver '70]
- Δ^1_2-Ramseyness is independent of ZFC.
  - Existence of a measurable cardinal implies Σ^1_2-Ramseyness.
  - $V = L$ implies $\neg(\Delta^1_2$-Ramseyness).
- There is $P \subseteq 2^{\mathbb{N}}$ which is not Ramsey. (Uses Axiom of Choice)

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Infinite game

Given $G \subseteq \mathbb{N}^{\mathbb{N}}$, consider the following *infinite game*:

I	a_0		a_2		$\cdots$
II		a_1		a_3	$\cdots$

$\rightarrow x = (a_0, a_1, a_2, a_3, \dots)$

The player I wins this game if $x \in G$; the player II wins if $x \notin G$.

A *strategy* for I (II resp.) is a function such that, for each step, input is every II (I)'s choice, output is a unique I (II)'s choice.

A strategy σ for I (II) is *winning*, if I (II) always wins no matter how II (I) plays, whenever I (II) follows σ.

$G \subseteq \mathbb{N}^{\mathbb{N}}$ is *determined* if either I or II has a winning strategy in this game.

Axiom of Choice implies "$\exists G$ (G is not determined)."

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Open games are determined

Theorem

Every open game (i.e. game where $G \subseteq \mathbb{N}^{\mathbb{N}}$ is open) is determined.

Proof.

Assume G is open and the player I does not have a winning strategy.
Then we can see that, for every play a_0 by I, there exists a play a_1 by II, such that I does not have a winning strategy after that.
Then, after that, for every play a_2 by I, there exists a play a_3 by II, such that I does not have a winning strategy after that.
This procedure gives a strategy for II, and since G is open this strategy is winning. □

Determinacy on each class

"Γ game" is a game of which winning set is Γ.

- Every open(Σ^0_1) game is determined. [Gale–Stewart '53]
- Every Borel(Δ^1_1) game is determined. (Needs Powerset $\times\ \aleph_1$ times) [Martin '75]
- Σ^1_1-determinacy is independent of ZFC.
  - If $\forall x \exists x^\sharp$ then every Σ^1_1 game is determined. [Martin, Harrington]
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Reverse Mathematics

Second-order arithmetic is the system which treats natural numbers and sets of natural numbers.
An axiom system of second-order arithmetic (*subsystem of second-order arithmetic*) typically consists of:

- Basic axioms of arithmetic (e.g. $x + y = y + x$)
- Induction scheme
- Set existence axiom (e.g. "every computable set exists.")

Reverse Mathematics is a program to find, given a theorem φ of mathematics, the smallest axiom which proves φ in second-order arithmetic.
E.g. the *Bolzano–Weierstraß theorem* (every bounded monotone sequence of real numbers converges) is equivalent to ACA_0 over RCA_0.
$\mathsf{RCA}_0 < \mathsf{WKL}_0 < \mathsf{ACA}_0 < \mathsf{ATR}_0 < \Pi^1_1\text{-}\mathsf{CA}_0$ (Big Five)

$< \Pi^1_1\text{-}\mathsf{TR}_0 < \Sigma^1_1\text{-}\mathsf{ID}_0 < \Pi^1_2\text{-}\mathsf{CA}_0 < \cdots < \mathsf{Z}_2$

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$$\mathsf{RCA}_0 < \mathsf{WKL}_0 < \mathsf{ACA}_0 < \mathsf{ATR}_0 < \Pi^1_1\text{-}\mathsf{CA}_0 \quad \text{(Big Five)}$$
$$< \Pi^1_1\text{-}\mathsf{TR}_0 < \Sigma^1_1\text{-}\mathsf{ID}_0 < \Pi^1_2\text{-}\mathsf{CA}_0 < \cdots < \mathsf{Z}_2$$

Ramseyness in second-order arithmetic

Theorem (RCA_0)

- $\mathsf{ATR}_0 \Leftrightarrow \Delta^0_1$-Ram $\Leftrightarrow \Sigma^0_1$-Ram. [Friedman–McAloon–Simpson '82]
- Π^1_1-$\mathsf{CA}_0 \Leftrightarrow \Delta^0_2$-Ram $\Leftrightarrow \Sigma^0_\infty$-Ram. [Simpson, Solovay]
- Π^1_1-$\mathsf{TR}_0 \Leftrightarrow \Delta^1_1$-Ram. [Tanaka '89]
- Σ^1_1-$\mathsf{ID}_0 \Leftrightarrow \Sigma^1_1$-Ram. [Tanaka '89]

ATR_0	$\leftrightarrow$	Σ^0_1-Ram
Π^1_1-CA_0	$\leftrightarrow$	Δ^0_2-Ram
Π^1_1-TR_0	$\leftrightarrow$	Δ^1_1-Ram
Σ^1_1-ID_0	$\leftrightarrow$	Σ^1_1-Ram
ZFC	$\nvdash$	Δ^1_2-Ram

Determinacy in second-order arithmetic

Theorem (RCA_0)

- $ATR_0 \Leftrightarrow \Delta^0_1$-Det $\Leftrightarrow \Sigma^0_1$-Det. [Steel '78]
- Π^1_1-$CA_0 \Leftrightarrow (\Sigma^0_1 \wedge \Pi^0_1)$-Det. [Tanaka '90]
- Π^1_1-$TR_0 \Leftrightarrow \Delta^0_2$-Det. [Tanaka '91]
- Σ^1_1-$ID_0 \Leftrightarrow \Sigma^0_2$-Det. [Tanaka '91]
- $[\Sigma^1_1]^{TR}$-$ID_0 \Leftrightarrow \Delta^0_3$-Det (over Π^1_3-TI_0). [MedSalem–Tanaka '08]
- Π^1_3-$CA_0 \Rightarrow \Sigma^0_3$-Det. [Welch '09]

(Note: Determinacy here is the determinacy of games over $\mathbb{N}$.)

Ramseyness and determinacy

ATR_0	$\leftrightarrow$	Σ^0_1-Ram	$\leftrightarrow$	Σ^0_1-Det
Π^1_1-CA_0	$\leftrightarrow$	Δ^0_2-Ram	$\leftrightarrow$	$\Sigma^0_1 \wedge \Pi^0_1$-Det
Π^1_1-TR_0	$\leftrightarrow$	Δ^1_1-Ram	$\leftrightarrow$	Δ^0_2-Det
Σ^1_1-ID_0	$\leftrightarrow$	Σ^1_1-Ram	$\leftrightarrow$	Σ^0_2-Det
$[\Sigma^1_1]^{\mathsf{TR}}$-$\mathsf{ID}_0$			$\leftrightarrow$	Δ^0_3-Det
Π^1_3-CA_0			$\vdash$	Σ^0_3-Det

over RCA_0 ($+\Pi^1_3$-TI_0)

Z_2			$\nvdash$	Σ^0_4-Det
ZFC			$\vdash$	Δ^1_1-Det

		Δ^1_2-Ram	Σ^1_1-Det
ZFC	$\nvdash$	Σ^1_2-Ram	Δ^1_2-Det
		$\vdots$	$\vdots$

Determinacy implies Ramsey property (1)

Theorem (Kastanas) (ZF + DC)

Let Γ be a class of subsets of $\mathbb{N}^{\mathbb{N}}$ (e.g. Σ^0_1, Σ^1_1, etc.). Then,

"the determinacy of Γ-games over reals"

implies

"the Ramsey property for sets of reals in Γ."

(Corollary: Every open set is Ramsey.)

This is proved by constructing certain game whose winning strategy implies Ramsey property (next 2 slides).

Γ-determinacy over reals $\Rightarrow$ Γ-Ramseyness

Given $P \subseteq 2^{\mathbb{N}}$ in Γ, consider the following game:

I	A_0		A_1		$\cdots$
II		(n_0, B_0)		(n_1, B_1)	$\cdots$

where $\mathbb{N} \supseteq A_i \supseteq B_i \supseteq A_{i+1}$: infinite, $n_i \in A_i$, $n_i < \min B_i$.

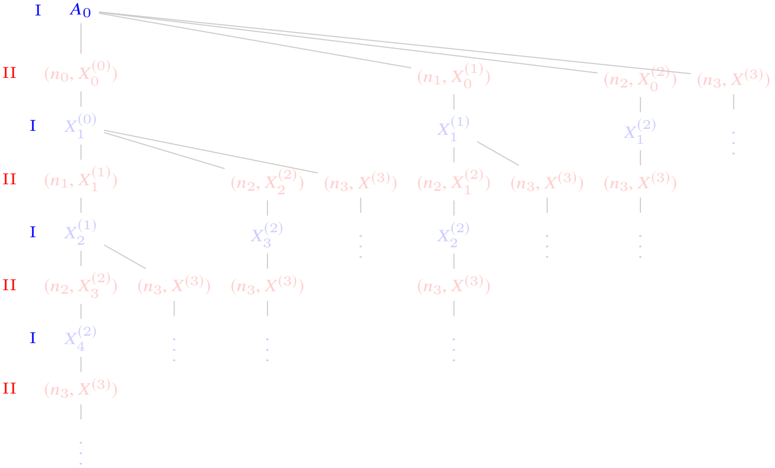
I wins if $\{ n_0, n_1, \dots \} \in P$.
This is a Γ-game.

Lemma

- I has a winning strategy $\Rightarrow \exists H \underset{\text{inf.}}{\subseteq} \mathbb{N}\ \forall X \underset{\text{inf.}}{\subseteq} H\ (X \in P)$.
- II has a winning strategy $\Rightarrow \exists H \underset{\text{inf.}}{\subseteq} \mathbb{N}\ \forall X \underset{\text{inf.}}{\subseteq} H\ (X \notin P)$.

σ: winning strategy for I.

Goal: Construct homogeneous set $H = \{ n_0 < n_1 < n_2 < n_3 < \cdots \}$.



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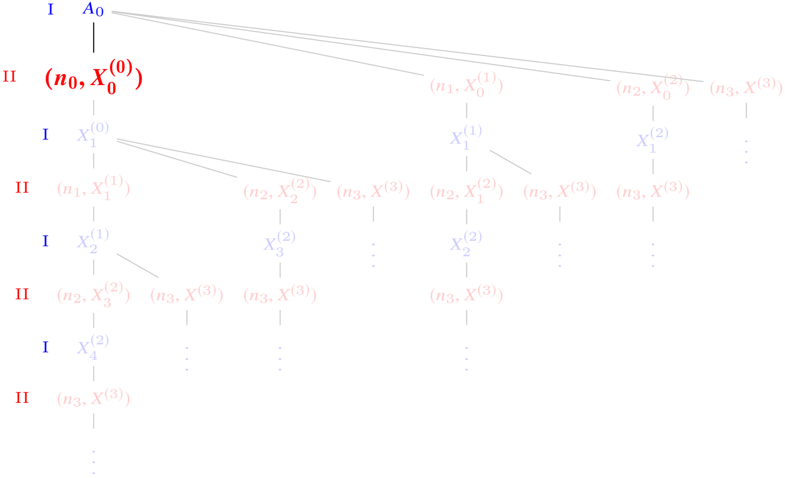

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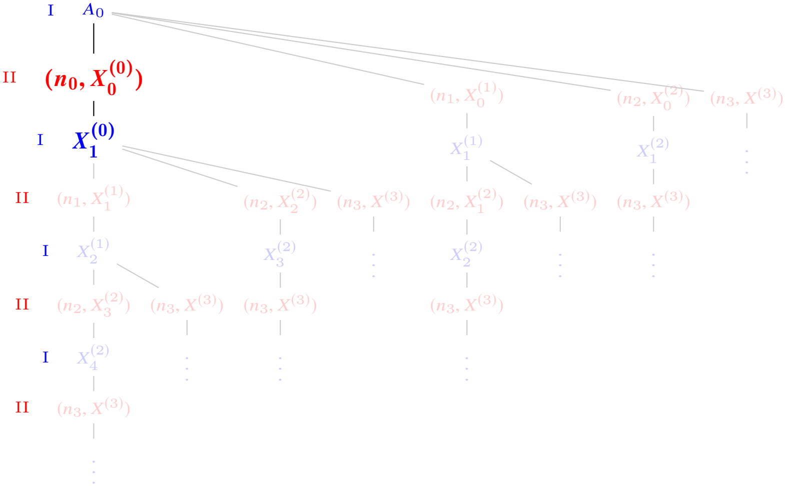

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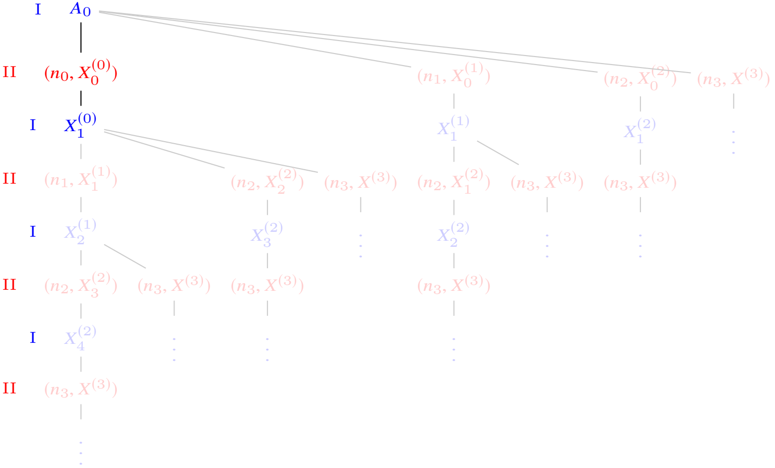

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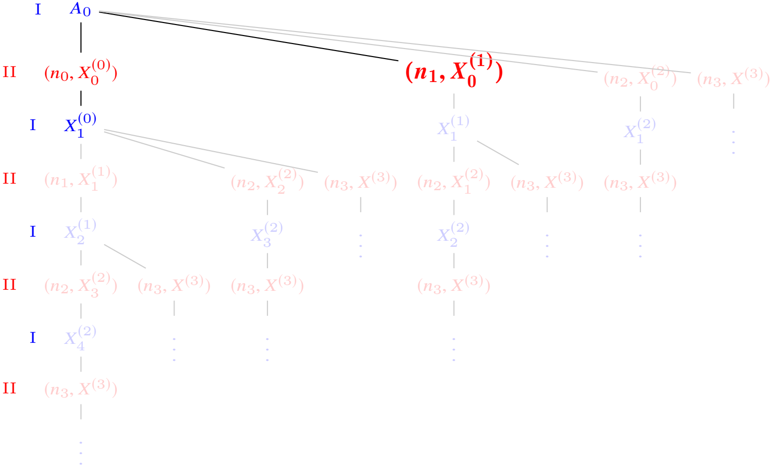

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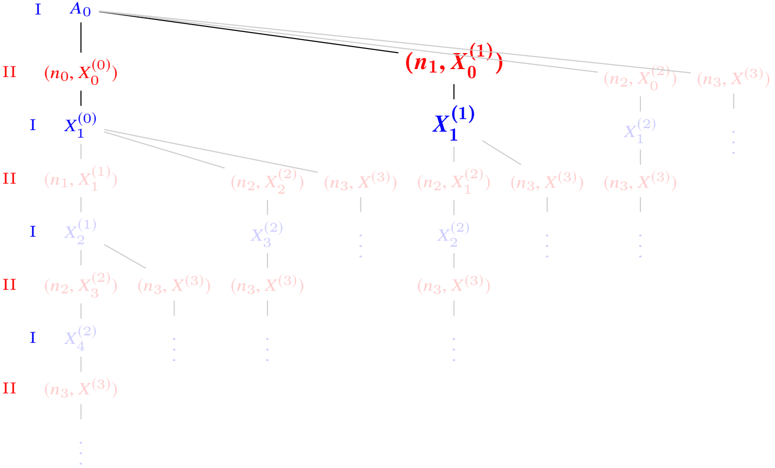

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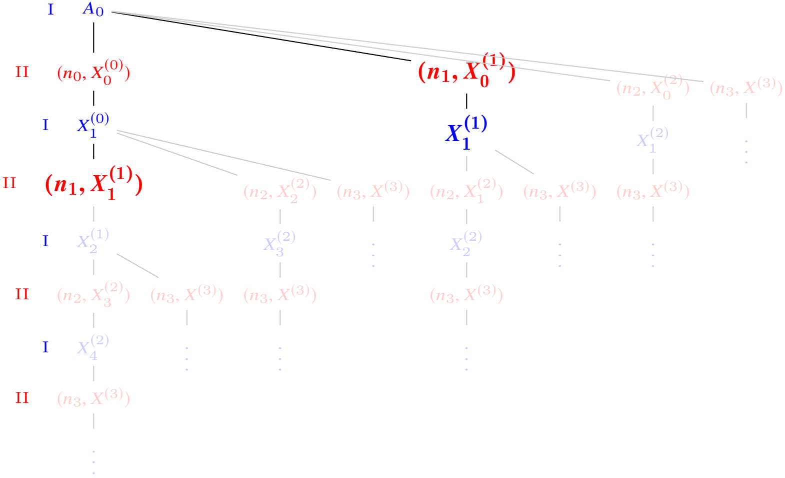

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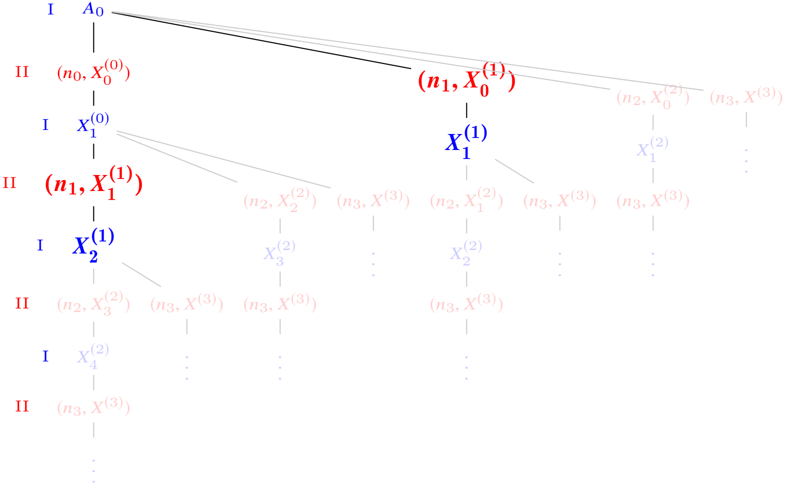

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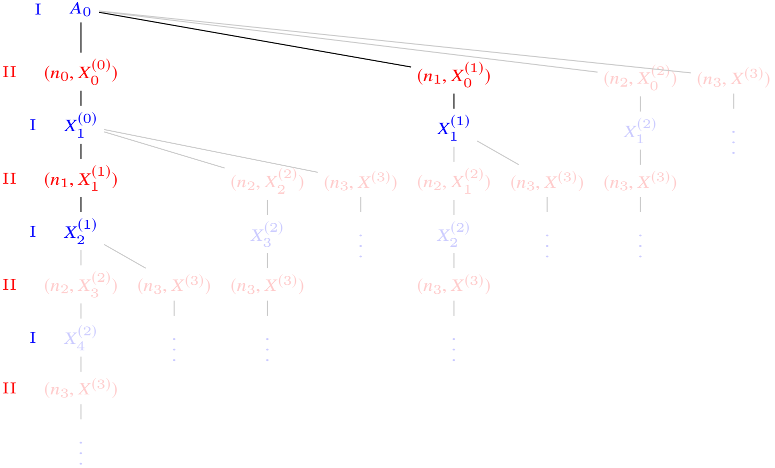

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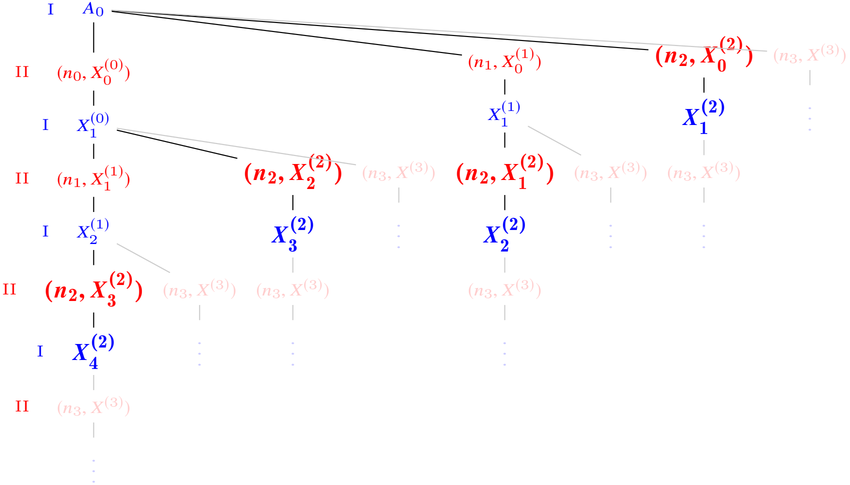

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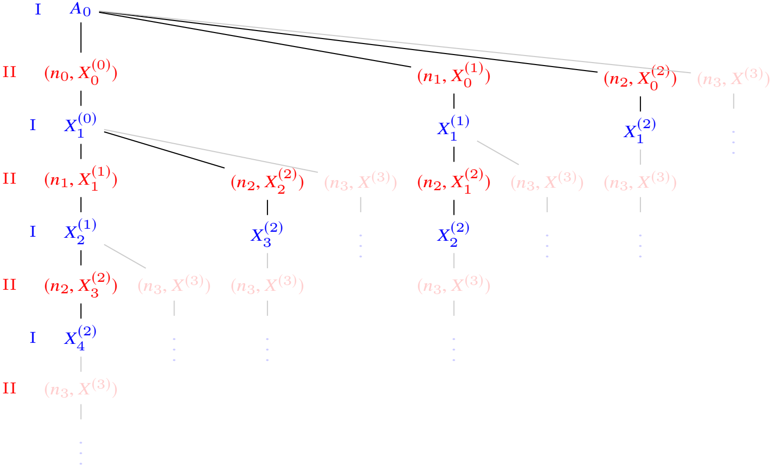

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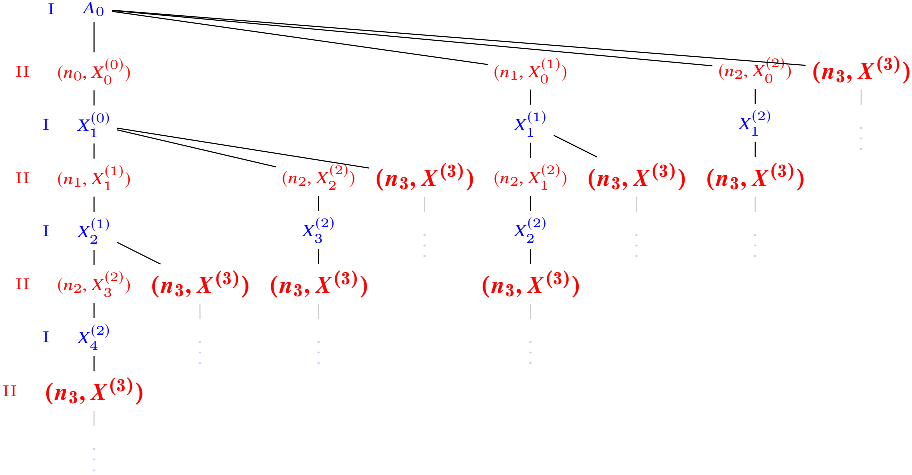

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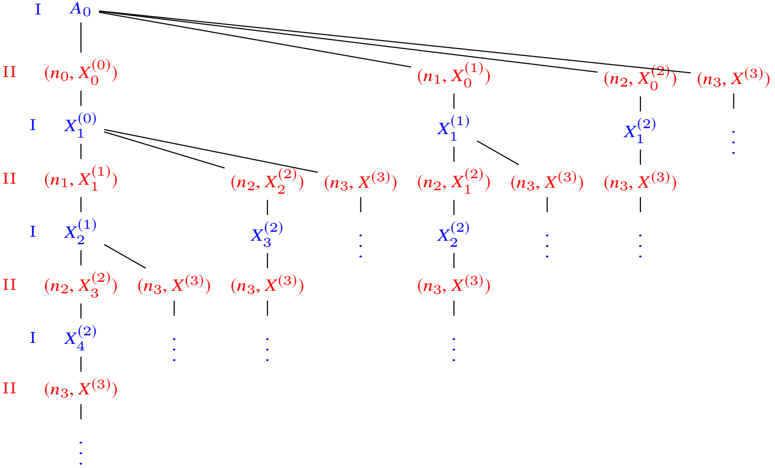

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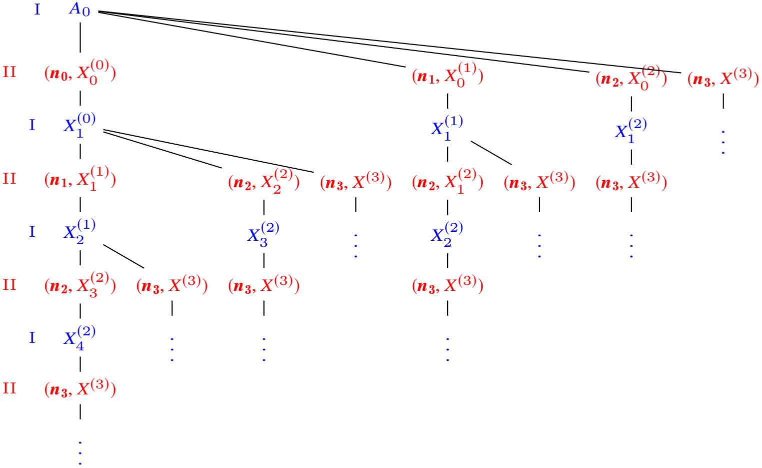

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Theorem (Kastanas) (ZF + DC)

Let Γ be a class of subsets of $\mathbb{N}^{\mathbb{N}}$ (e.g. Σ^0_1, Σ^1_1, etc.). Then,

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Summary and Question

ATR_0	$\leftrightarrow$	Σ^0_1-Ram	$\leftrightarrow$	Σ^0_1-Det
Π^1_1-CA_0	$\leftrightarrow$	Δ^0_2-Ram	$\leftrightarrow$	$\Sigma^0_1 \wedge \Pi^0_1$-Det
Π^1_1-TR_0	$\leftrightarrow$	Δ^1_1-Ram	$\leftrightarrow$	Δ^0_2-Det
Σ^1_1-ID_0	$\leftrightarrow$	Σ^1_1-Ram	$\leftrightarrow$	Σ^0_2-Det
$[\Sigma^1_1]^{\mathsf{TR}}$-$\mathsf{ID}_0$			$\leftrightarrow$	Δ^0_3-Det
Π^1_3-CA_0			$\vdash$	Σ^0_3-Det

over RCA_0 ($+\Pi^1_3$-TI_0)

Z_2			$\nvdash$	Σ^0_4-Det
ZFC			$\vdash$	Δ^1_1-Det

		Σ^1_2-Ram	$\leftarrow$	Σ^1_1-Det
ZFC	$\nvdash$	Σ^1_3-Ram	$\leftarrow$	Σ^1_2-Det
		$\vdots$		$\vdots$

over ZFC

Conjecture

For all $n \in \omega$, $\mathsf{RCA}_0 ? \vdash \Sigma^1_n\text{-Det} \rightarrow \Sigma^1_{n+1}\text{-Ram}.$

Thank you very much.

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